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ABSTRACT

This is a guide for use in semester-long courses in Elementary Functions and Analytic Geometry. A list of entry-level skills and a list of approved textbooks is provided. Each of the 18 units consists of: (1) overview, suggestions for teachers, and suggested time; (2) list of objectives; (3) cross-references guide to approved textbooks; (4) sample problems for each objective; and (5) answer key for sample problems. For Elementary Functions, the nine units are: (1) introduction to functions, (2) algebraic functions, (3) exponential and logarithmic functions, (4) circular functions, (5) trigonometric identities, (6) inverse circular functions and trigonometric equations, (7) applications of trigonometric functions, (8) complex numbers, and (9) functions on the natural numbers. For Analytic Geometry, the nine units are: (1) introduction to analytic geometry, (2) points, lines, and planes in space, (3) vectors in a plane, (4) vectors in space, (5) conic sections, (6) matrices, (7) polar coordinates, (8) parametric representation of curves, and (9) surfaces. (MNS)

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# PRE-CALCULUS INSTRUCTIONAL GUIDE

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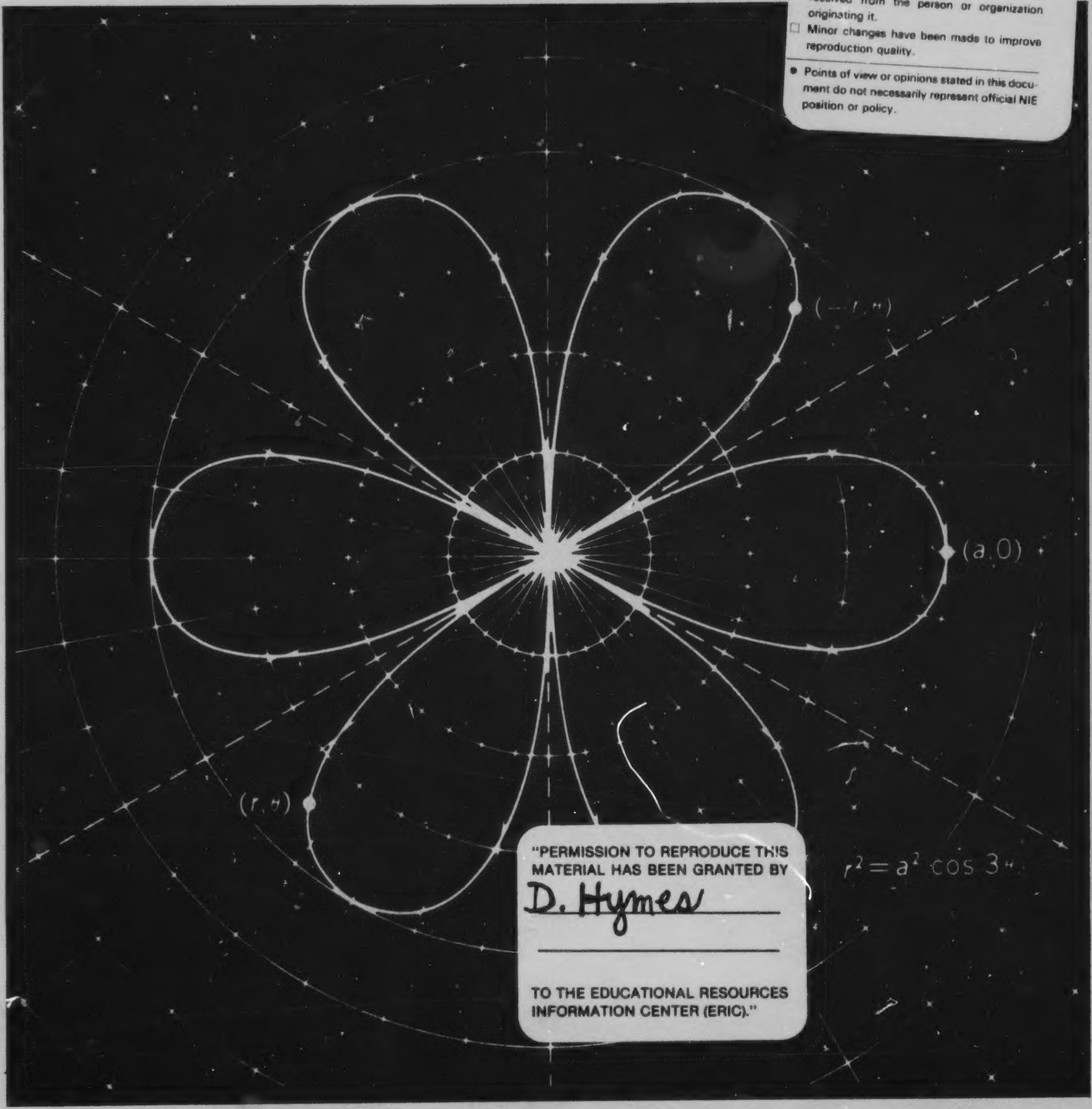

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SE041277

# PRE-CALCULUS

# INSTRUCTIONAL GUIDE

# for

# ELEMENTARY FUNCTIONS

# ANALYTIC GEOMETRY

Montgomery County Public Schools
Rockville, Maryland

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Joan Burks, Woodward Senior High School
Richard Miller, Whitman Senior High School
Elaine Schmitt, Wheaton Senior High School

## General Introductory Comments

This is a guide for use in Elementary Functions and Analytic Geometry, semester courses which, combined, give a year of preparation for Calculus.

The intent of these courses is to clarify and extend the student's elementary understanding of functions and to emphasize functions as the major unifying idea of all mathematics.

Several of the important ideas that frequently appear in the courses are:

1. classification of the types of functions
2. identification of their properties
3. construction of their graphs
4. determination of their equations
5. applications

Inasmuch as this year is a prerequisite for calculus, the student should be informed that it will take diligent effort on his or her part to meet the challenge and rigor of this material. In addition, proficiency in algebra skills and some background in trigonometry are considered to be indispensable for success in this subject.

A list of entry level skills from Algebra 2 and Trigonometry is provided. This list represents minimal skills needed throughout the course. It is recommended that the student be given a copy of this list.

The order of units in this instructional guide for the first semester may not necessarily conform to the order of topics found in your textbook. The sequence of units may be varied somewhat, with one arrangement being Units I, II, III, IV, VII, V, VI, VIII, and IX.

## Organization of Individual Units

Each unit will consist of the following:

1. Overview, suggestions to the teacher, and suggested time
2. List of objectives
3. Cross-references guide to approved textbooks
4. Sample problems for each objective
5. Answer key for sample problems

Throughout the units, several objectives will be marked **. These objectives are more comprehensive because they combine previous objectives without introducing any new material.

It is suggested that at a minimum the following textbooks be available for reference or classroom use:

Shanks, Pre-Calculus Mathematics

Crosswhite, Pre-Calculus Mathematics

Coxford, Advanced Mathematics

The individual time allocations include testing as well as instructional days. Five days are allowed during the first semester for class periods lost due to various circumstances.

## Entry Level Skills for Pre-Calculus

1. Graph linear and quadratic functions and relations.
2. Determine the distance between two points.
3. Perform the four fundamental operations with complex numbers.
4. Factor algebraic expressions.
5. Perform the four fundamental operations with rational expressions.
6. Solve quadratic equations over the set of complex numbers.
7. Solve systems of equations.
8. Sketch the graph of polynomial functions of the form $y = x^n$ where n is a positive integer.
9. Apply the Laws of Exponents to simple expressions.
10. Use the logarithmic tables to determine the logarithm of a number and its anti-log.
11. Solve simple logarithmic and exponential equations.
12. Sketch functions of the form $y = a^x$ where a is a rational number.
13. Determine equations of lines and conic sections, given pertinent information.
14. Determine the domain of a given function or relation.
15. State the definitions of the $\sin\theta$ and $\cos\theta$ functions in terms of the unit circle.
16. Determine the trigonometric functional values of the integral multiples of $30^\circ$ and $45^\circ$ ($\frac{\pi}{6}$, $\frac{\pi}{4}$ multiples).
17. Determine the angle (multiples of $30^\circ$ and $45^\circ$) whose functional value has been given.
18. Evaluate functional values of angles, using a table.
19. Solve simple trigonometric equations.
20. Apply the fundamental trigonometric relations to verify identities.

Entry Level Skills for Pre-Calculus (continued)

21. State the sum and difference formulas for the sinθ and cosθ.

22. State the double and half-angle formulas for sinθ and cosθ.

23. Graph the six trigonometric functions.

24. Apply the trigonometric functions to the solution of right triangles.

25. State the Law of Sines and Law of Cosines.

NOTE: Some of the entry level skills related to trigonometry may be repeated in Pre-Calculus

## Approved Textbooks

Coxford, Arthur F., et al. *Advanced Mathematics - A Preparation for Calculus*. second edition. New York: Harcourt Brace Jovanovich, 1978.

Crosswhite, F. Joe, et al. *Pre-Calculus Mathematics*. Columbus, Ohio: Charles E. Merrill, 1976.

Fuller, Gordon. *Analytic Geometry*. Fifth ed. Menlo Park, Calif.: Addison Wesley, 1979

Shanks, Merrill E., et al. *Pre-Calculus Mathematics*. Menlo Park, Calif.: Addison Wesley, 1981

Sorgenfrey, Robert H., et al. *Analysis of Elementary Functions*. Boston: Houghton Mifflin, 1980.

Wooton, William, et al. Modern Analytic Geometry. Boston: Houghton Mifflin, 1978.

## References

Dolciani, Mary, et al. *Algebra and Trigonometry, Structure and Method*. new edition. Boston: Houghton Mifflin, 1982.

Foerster, Paul A. *Trigonometry: Functions and Applications*. Menlo Park, Calif.: Addison-Wesley, 1977.

Protter, Murray H., et al. *Analytic Geometry*. Menlo Park, Calif.: Addison-Wesley, 1975.

Rees, Paul K., et al. *Algebra, Trigonometry, and Analytic Geometry*. New York: McGraw-Hill, 1975.

## Outline of Course Content and Time Allocation

### First Semester - Elementary Functions

| | Unit | Number of Days |
|---|---|---|
| I. | Introduction to Functions | 12 |
| II. | Algebraic Functions | 10 |
| III. | Exponential and Logarithmic Functions | 8 |
| IV. | Circular Functions | 12 |
| V. | Trigonometric Identities | 10 |
| VI. | Inverse Circular Functions and Trigonometric Equations | 10 |
| VII. | Applications of Trigonometric Functions | 10 |
| VIII. | Complex Numbers | 3 |
| IX. | Functions on the Natural Numbers | 10 |
| | | TOTAL 85 days |

### Second Semester - Analytic Geometry

| | Unit | Number of Days |
|---|---|---|
| X. | Introduction to Analytic Geometry | 6 |
| IX. | Points, Lines and Planes in Space | 10 |
| XII. | Vectors in a Plane | 7 |
| XIII. | Vectors in Space | 9 |
| XIV. | Conic Sections | 16 |
| XV. | Matrices | 10 |
| XVI. | Polar Coordinates | 7 |
| XVII. | Parametric Representation of Curves | 10 |
| XVIII. | Surfaces | 10 |
| | | TOTAL 85 days |

Pre-Calculus - Elementary Functions

## Unit I Introduction to Functions

### Overview

This unit provides the student with the basic principles of functions. It deals with what a function is, how functions may be combined, and their characteristics and properties. It is as vital to the future calculus student as any other unit in this guide. Virtually all of the concepts, relationships, and vocabulary in this unit will be applied in subsequent units, not to mention the calculus course itself.

Because of the scattering of the topics throughout the various approved tests, the teacher may choose to alter the sequence of objectives. The student should, nevertheless, be able to perform all of the objectives at some point in the course.

### Suggestions to the Teacher

No single approved text seems best for this unit. Several sources may be required. Some comments regarding some of the approved texts follow:

Pre-Calculus Mathematics (Shanks) - Mostly adequate, sequence fair

Pre-Calculus Mathematics (Crosswhite) - Mostly adequate, sequence jumbled

Analysis of Elementary Functions (Sorgenfrey) - A few gaps, strong in objectives 1-8

Suggested time: 12 days

Pre-Calculus

## Unit I Introduction to Functions

### PERFORMANCE OBJECTIVES

1. State the definition of a function.
2. Determine whether a given relation is a function.
3. Identify the domain and range of a given function.
4. State the meaning of the symbols used in functional notation.
5. Determine whether a given function is one-to-one.
6. Given the two functions f and g, determine the formulas for functions $f + g$ and $f - g$ and sketch them.
7. Given two functions f and g, determine the formula for the functions $f \bullet g$ and $\frac{f}{g}$, $g \neq 0$.
8. Given two functions f and g, determine the formulas for the composite functions $f \circ g$ and $g \circ f$.
9. Sketch the graph of special functions (step, greatest integer, signum, and absolute value).
10. Classify a function as being increasing, decreasing, or neither over a given interval.
11. Classify a function over a given interval as being bounded, bounded above, bounded below, or unbounded.
12. Classify a function as being continuous or not continuous at a given point or over a given interval.
13. Determine whether a given relation is symmetric to the y-axis, x-axis, or origin.
14. Determine whether a given function is odd, even, or neither.
15. Determine the inverse of a given function.
16. Graph a given function and its inverse on the same set of axes.
17. Modify the domain of a given function so that its inverse is a function.
18. Graph $f(x)$ and $f^{-1}(x)$ on the same set of axes. (Make $f(x)$ one-to-one if necessary.)
19. Demonstrate that the properties of the real number field apply to the set of functions under the operations of addition and multiplication.

Pre-Calculus

Unit I    Introduction to Functions

Optional

20. Determine which properties of a field apply to the composition of functions operation.

Unit I - Introduction of Functions

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
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CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
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| 18 | 48 | 83-88 | | 31-35 | 40-46 | |
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**I-1 State the definition of a function.**

1. Define a function in terms of ordered pairs (x, y).
2. Define a function in terms of mapping.

**I-2 Determine whether a given relation is a function.**

1. Indicate which of the following relations are functions:

   a) $3y^2 - x = 4$

   b) $x^2 - y = 1$

   c) $y = -|x + 3|$

   d) $x^2 + y^2 = 8$

2. Which of the following sets are functions?

   a) $\{(x, y) : y = x\}$

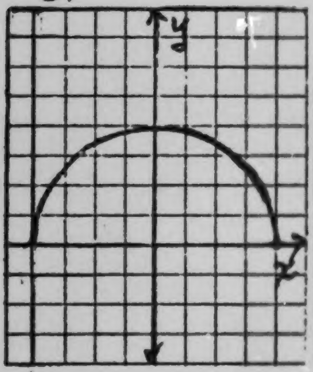
   b) $\{(x, y) : y = -5\}$

   c) $\{(x, y) : y = \sqrt{x}\}$

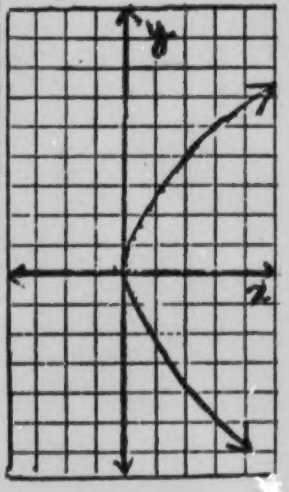
   d) $\{(x, y) : y = xy\}$

3. State which of the graphs of relations given below are functions.

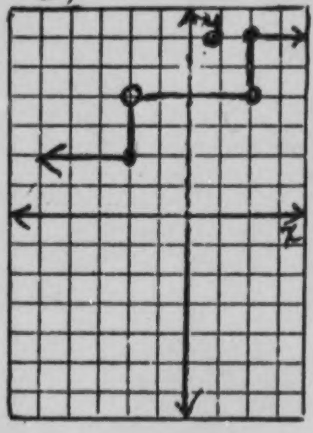
a)



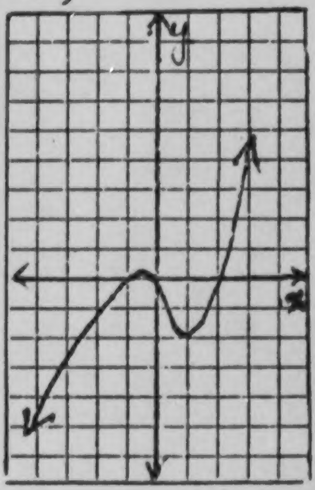
b)



c)



d)



**I-3 Identify the domain and range of a given function.**

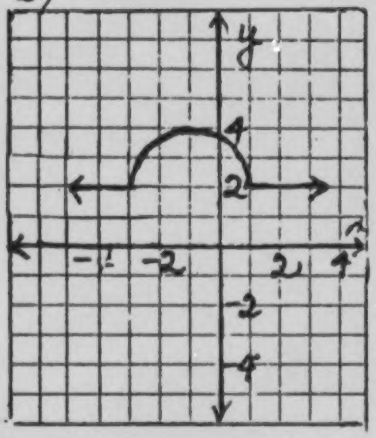
1. Identify the domain and range of $f(x) = x^2 - 2$.

2. Identify the domain and range of $g(x) = |x| + 3$.

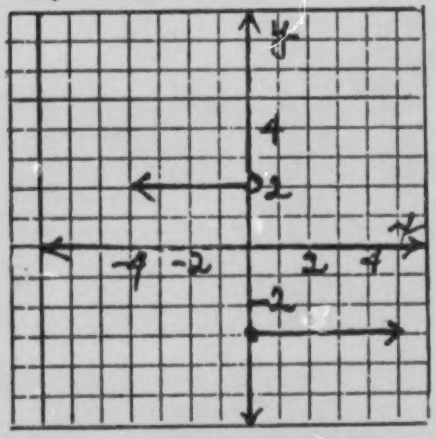
3. Identify the domain and range of $h(x) = \sqrt{9 - x^2}$.

4. Determine the domain and range of each function whose graph is given below.

a)



b)



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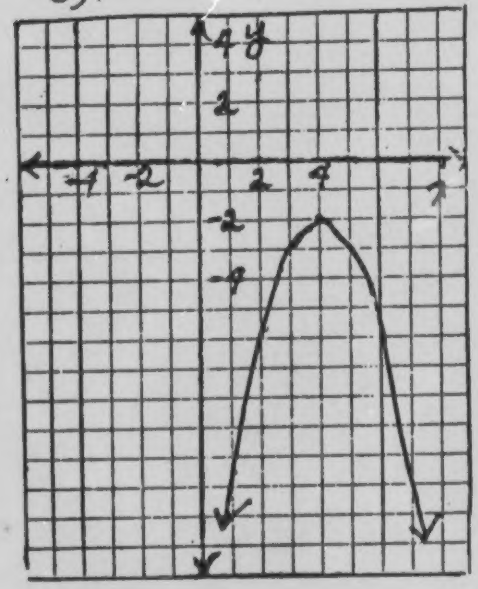


I-4 State the meaning of the symbols used in functional notation.

1. State the exact and complete meaning of "$f(x) = y$."

2. State the exact and complete meaning of "$f: x \longrightarrow y$."

3. Translate completely into words: "$g(3) = 7$."

4. Translate completely into words: "$F: x \longrightarrow x^2 - 4$."

**I-5 Determine whether a given function is one-to-one.**

1. Determine whether or not each of the following functions is one-to one.

   a) $y = x$

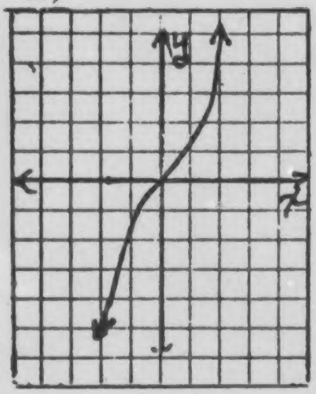
   b) $y = x^2$

   c) $y = x^3 - 4x$

   d) $y = \frac{1}{x}$

2. Determine whether or not each of the following functions is one-to-one.

a)



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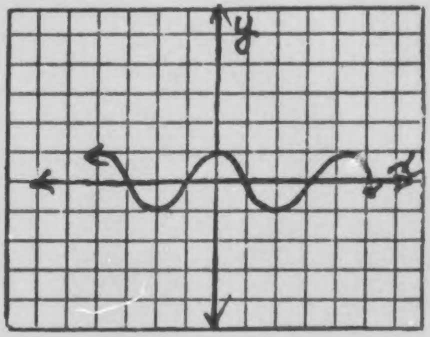

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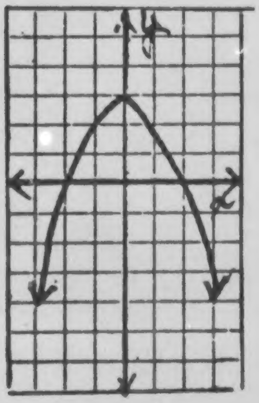

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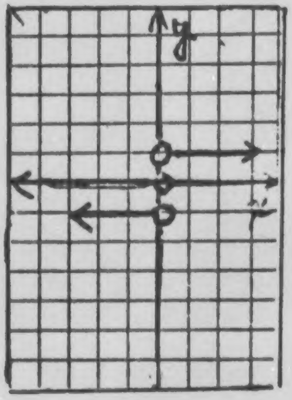

I-6 Given two functions f and g, determine the formulas for the functions f + g and f - g and sketch them.

1. If $f(x) = x^2 + x + 1$ and $g(x) = x$, determine a formula for $(f + g)(x)$ and $(f - g)(x)$. Sketch the new functions.

2. If $f(x) = - x^2 + 5$ and $g(x) = x^2 - 4$, determine a formula for $(f + g)(x)$ and $(f - g)(x)$. Sketch the new functions.

3. Given $f(x) = \frac{1}{x}$ and $g(x) = 1$, determine formulas for two new functions $f + g$ and $f - g$. Sketch the new functions.

4. Given $f(x) = |x - 1|$ and $g(x\ ) = (x)$ for all $x \geq 1$, determine formulas for two new functions $f + g$ and $f - g$. Sketch the new functions.

I-7 Given two functions f and g, determine the formulas* for the functions $f \bullet g$ and $\frac{f}{g}$, $g \neq 0$.

1. If $f(x) = x^2$ and $g(x) = \frac{1}{2x}$ $(x \neq 0)$, determine a formula for $(f \cdot g)(x)$ and $\frac{f}{g}(x)$.

2. Given $f(x) = x$ and $g(x) = 2 - x$ for all $x \geq 0$, determine the formula for two new functions $(f \cdot g)(x)$ and $\frac{f}{g}(x)$.

3. If $f(x) = x - 1$ and $g(x) = x + 2$, determine a formula for $(f \cdot g)(x)$ and $\frac{f}{g}(x)$.

---

*Domains, ranges, and/or graphs may be substituted or added to the problem's requirements.

I-8 Given two functions f and g, determine the formulas* for the composite functions $f \circ g$ and $g \circ f$.

1. If $f(x) = x^2$ and $g(x) = \sqrt{x - 4}$, determine a formula for $f \circ g$ and $g \circ f$.

2. If $f(x) - 3x + 2$ and $g(x) = \frac{1}{3}x - \frac{2}{3}$, determine the formulas for $f \circ g$ and $g \circ f$.

3. Given $f(x) = x$ and $g(x) = \frac{1}{x^2 - 1}$, determine the formulas for $f \circ g$ and $g \circ f$.

4. Given $f(x) = x^3 - 8$ and $g(x) = x + 2$, determine the formulas for $f \circ g$ and $g \circ f$.

---

*Domains, ranges, and/or graphs may be substituted or added to the problems requirements.

I-9 Sketch the graph of special functions (step, greatest integer, signum, and absolute value).

1. Sketch the graph of the step function:

$$f(x) = \begin{cases} 3 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$$

2. Sketch the graph of the greatest integer function:

$$f(x) = [\![x + \tfrac{1}{2}]\!]$$

3. Sketch the graph of the absolute value function:

$$f(x) = 2\,|x + 3| - 1$$

4. Sketch the graph of the signum function:

$f(x) = \dfrac{x - 4}{\left|\frac{1}{2}x - 2\right|}$ . (Hint: first factor out $\frac{1}{2}$ from the denominator.)

5. Sketch the graph of the periodic function $f(x) = \sin\left(x + \frac{\pi}{2}\right)$.

**I-10 Classify a function as being increasing, decreasing, or neither over a given interval.**

1. Classify the following intervals of the function $f(x) = (x - 3)$ as being increasing, decreasing, or neither:

   a) $[-1, 1]$

   b) $\langle 1, 4 \rangle$

   c) $\langle 3, 5]$

2. Classify the following intervals of the function $f(x) = \cos x$ as being increasing, decreasing, or neither.

   a) $\left\langle \frac{\pi}{2}, \frac{\pi}{2} \right\rangle$

   b) $\left\langle \pi, \frac{5\pi}{4} \right\rangle$

   c) $\left[ \frac{\pi}{4}, \frac{3\pi}{4} \right]$

3. Classify the following intervals of the function $f(x) = \frac{1}{x + 2}$ as being increasing, decreasing, or neither.

   a) $[-4, -2\rangle$

   b) $\langle -2, 0 \rangle$

   c) $[0, \infty\rangle$

4. For $x \in \langle 0, 5]$ classify the following functions as being increasing, decreasing, or neither:

   a) $f(x) = \frac{1}{x}$

   c) $h(x) = [x]$

   b) $g(x) = \sqrt{25 - x^2}$

   d) $k(x) = |x - 3|$

I-11 Classify a function over a given interval as being bounded, bounded above, bounded below, or unbounded.

1. Classify the following intervals of the function $f(x) = 2^x$ as being bounded, bounded above, bounded below, or unbounded.

   a) $x \epsilon \{\text{negative reals}\}$

   b) $x \epsilon \{\text{positive reals}\}$

   c) $[-1, 1]$

2. Classify the following intervals of the function $f(x) = -|x + 3|$ as being bounded, bounded above, bounded below, or unbounded.

   a) $x \epsilon \{\text{negative reals}\}$

   b) $x \epsilon \{\text{positive reals}\}$

   c) $[-3, 3]$

3. Classify the following intervals of the function $f(x) = x^3 - x$ as being bounded, bounded above, bounded below, or unbounded.

   a) $\langle -1, 0 \rangle$

   b) $\langle 0, 1 \rangle$

   c) $\langle 1, \infty \rangle$

4. For $x \epsilon \{\text{negative reals}\}$ classify the following functions as being bounded, bounded above, bounded below, or unbounded.

   a) $f(x) = [x]$

   b) $g(x) = \frac{1}{x}$

   c) $h(x) = \tan x$

   d) $k(x) = (x + 100)^3$

**I-12** Classify a function as being continuous or not continuous at a given point or over a given interval.

1. Classify the following points of the function $f(x) = \frac{1}{x - 2}$ as being continuous or discontinuous.

   a) $x = 0$

   b) $x = 2$

   c) $x = -2$

2. At the point $x = 1$, classify the following functions as being continuous or discontinuous.

   a) $f(x) = \frac{x}{|x|}$

   b) $g(x) = [x]$

   c) $h(x) = \frac{x^2 - 1}{x - 1}$

   d) $k(x) = |x - 1|$

3. Classify the following intervals of the function $f(x) = \cot x$ as being continuous or not continuous.

   a) $\left[0, \frac{\pi}{2}\right]$

   b) $\left[\frac{\pi}{2}, \pi\right\rangle$

   c) $\left\langle\pi, 2\pi\right\rangle$

I-12 (continued)

4. For $x \in \langle 0, 5]$ classify the following functions as being continuous or not continuous.

   a) $f(x) = \dfrac{x^2 + 3x - 10}{x + 5}$

   b) $g(x) = \dfrac{1}{x}$

   c) $h(x) = |x^2 - 4|$

   d) $k(x) = \dfrac{x^2 - 25}{x - 5}$

I-13 Determine whether a given relation is symmetric to the y-axis, x-axis, or origin.

1. Determine whether the following relations are symmetric to the y-axis, x-axis, origin, or some combination of the above.

   a) $\{(x, y) : x^2 - y^2 = 4\}$

   b) $\{(x, y) : x^2 + y^2 = 9;\ y \le 0\}$

   c) $\{(x, y) : 4x^2 + y^2 = 4;\ x \ge 0\}$

   d) $\{(x, y) : x = -y^2\}$

2. Determine whether the following relations are symmetric to the y-axis, x-axis, origin, some combination of the above, or none of the above.

   a) $\{(x, y) : xy = -6\}$

   b) $\{(x, y) : x = |y|\}$

   c) $\{(x, y) : x = y^3\}$

   d) $\{(x, y) : y = x^3 + x + 1\}$

**I-14 Determine whether a given function is odd, even, or neither.**

1. Determine whether the following functions are odd, even, or neither.

   a) $f(x) = x^4 + 3$

   b) $g(x) = -x^3$

   c) $h(x) = \sin x$

   d) $k(x) = \cos x$

2. Determine whether the following functions are odd, even, or neither.

   a) $f(x) = |x - 3|$

   b) $g(x) = \frac{1}{x}$

   c) $h(x) = \tan x$

   d) $k(x) = \frac{x}{|x|}$

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**I-15 Determine the inverse of a given function.**

1. Determine the inverse of each of the following functions. (For each formula, write y in terms of x.)

   a) $y = x^2$

   b) $y = x^3$

   c) $y = \frac{1}{3}x + 2$

   d) $y = \frac{1}{x}$

2. Determine the inverse of each of the following functions. (For each formula, write y in terms of x.)

   a) $y = \sqrt{16 - x^2}$

   b) $y = \frac{x}{x - 1}$

   c) $2y = \sqrt{4 - x^2}$

**I-16 Graph a given function and its inverse on the same set of axes.**

1. Graph $f(x) = x^2 - 3$ and its inverse on the same set of axes.

2. Graph $f(x) = \frac{1}{2}x + 3$ and its inverse on the same set of axes.

3. Graph $f(x) = -\sqrt{x^2 - 4}$ and its inverse on the same set of axes.

4. Graph $f(x) = \sqrt{9 - x^2}$ ; $x \leq 0$ and its inverse on the same set of axes.

I-17 Modify the domain of a given function so that its inverse is a function.*

1. Restrict the domains of each of the functions given below so that their inverses will also be functions. State the modified domains.

   a) $f(x) = x^2 + 3;\ x \in$ Reals

   b) $f(x) = \sqrt{x^2 + 1};\ x \in$ Reals

   c) $f(x) = \sqrt{16 - x^2}\ ;\ |x| \leq 4$

   d) $f(x) = -\sqrt{x^2 - 16};\ |x| \geq 4$

   e) $f(x) = |x| - 2;\ x \in$ Reals

   f) $f(x) = \sqrt{x} + 4;\ x \geq 0$

---

* The modification referred to in this objective is to make the original function a one-to-one type to facilitate the formation or graph of the inverse function $f^{-1}(x)$.

I-18 Graph $f(x)$ and $f^{-1}(x)$ on the same set of axes. (Make $f(x)$ one-to-one, if necessary.)

1. Sketch the graph of $f(x)$ and $f^{-1}(x)$ for the functions given below. (Restrict the domain of $f(x)$ as necessary to graph it as a one-to-one function.)

   a) $f(x) = 3x - 1$

   b) $f(x) = -\sqrt{9 - x^2}$

   c) $f(x) = x^2 - 4x$

   d) $f(x) = |x| - x$

I-19 Demonstrate that the properties of the real number field apply to the set of functions under the operations of addition and multiplication.

1. Give an example for each of the field properties as applied to the set of functions and the operation of addition (closure, associativity, identity, inverse, commutativity).
2. Give an example for each of the field properties as applied to the set of functions and the operation of multiplication (closure, associativity, identity, inverse, commutativity).
3. Give an example of the distributive property, multiplication over addition, as it may apply to the set of functions.

I-20 Determine which properties of a field apply to the composition of functions operation.

1. Determine which properties of a field can be applied to the set of functions under the operation composition of functions. $(f \circ g)$
2. Give an example of each of the properties of a field that do apply to the set of functions under the operation composition of functions. $(f \circ g)$
3. Under what circumstances would the commutative property hold for the composition of functions operation? Give an example.

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## I-1

1. A function is a set of ordered pairs, $(x, y)$, such that for each $x \in X$ there is exactly one $y \in Y$.

   (alternative) A function is a relation in which for each first element there corresponds a unique second element.

2. A function f is a rule which assigns to each member $x \in X$ a unique member $y \in Y$. The function f is said to map X into Y.

## I-2

1. a) not f

   b) f

   c) f

   d) not f

2. a) f

   b) f

   c) f

   d) not f

3. a) f

   b) not f

   c) not f

   d) f

## I-3

1. D: $x \in$ Reals; R: $y \geq -2$

2. D: $x \in$ Reals; R: $y \geq 3$

3. D: $-3 \leq x \quad 3$; R: $0 \leq y \leq 3$

4. a) D: $x \in$ Reals; R: $2 \leq y \leq 4$
   b) D: $x \in$ Reals; R: $y = 2$ or $y = -3$
   c) D: $x \in$ Reals; R: $y \leq -2$

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## I-4

1. "The value of the f function at x is y."

2. "f is the function that maps x into y."

3. "The value of the g function at x = 3 is 7."

4. "f is the function which associates with each number x the number $x^2 - 4$."

## I-5

1. a) one-to-one
   b) not one-to-one
   c) not one-to-one
   d) one-to-one

2. a) one-to-one
   b) not one-to-one
   c) not one-to-one
   d) one-to-one

## I-6

1. $x^2 + 2x + 1$ or $(x + 1)^2$ ; $x^2 + 1$ (sketches not given)

2. 1; $-2x^2 + 9$

3. $\frac{1 + x}{x}$ ; $\frac{1 - x}{x}$

4. $2x - 1$; $-1$

## I-7

1. $\frac{1}{2}x$; $2x^3$ $(x \neq 0)$

2. $2x - x^2$; $\frac{x}{2 - x}$ $(x \neq 2)$

3. $x^2 + x - 2$; $\frac{x - 1}{x + 2}$ $(x \neq -2)$

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I-8

1. $x - 4$; $\sqrt{x^2 - 4}$

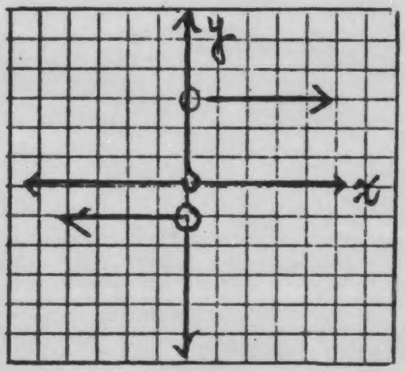
2. $x$ ; $x$

3. $\frac{1}{x^2 - 1}$ ; $\frac{1}{x^2 - 1}$

4. $x^3 + 6x^2 + 12x$; $x^3 - 6$

I-9

1.



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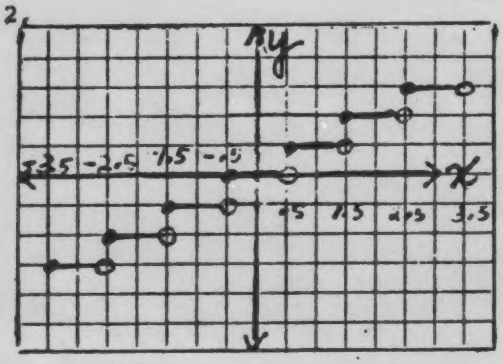
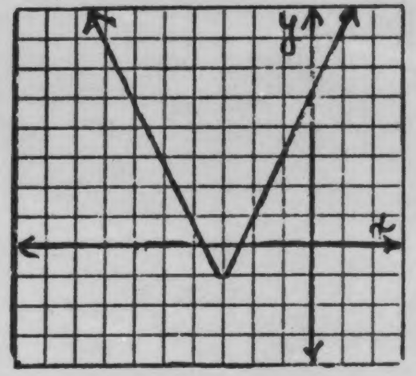


I-9 (continued)

3.



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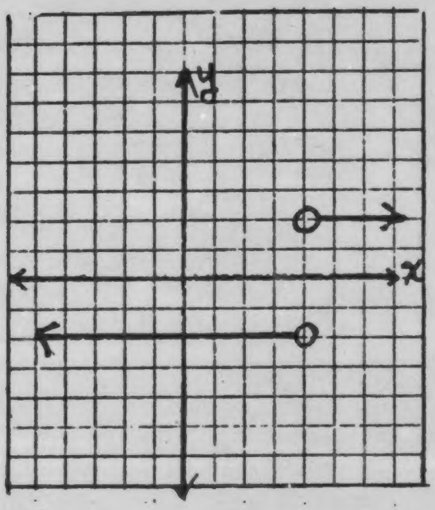

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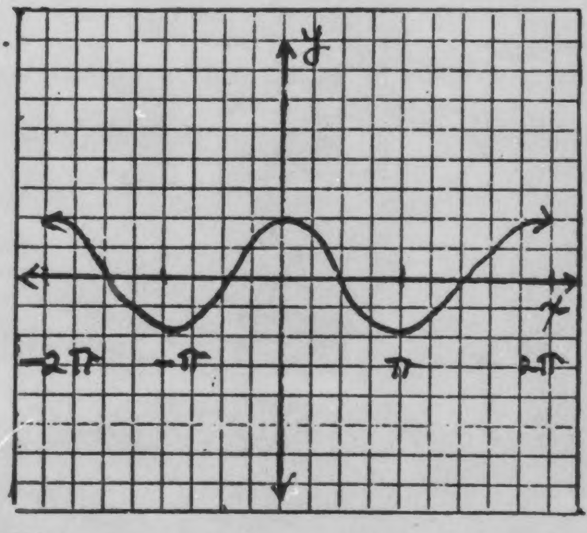

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## I-10

1. a) decrease b) neither c) increase
2. a) neither b) increase c) decrease
3. a) decrease b) decrease c) decrease
4. a) decrease b) decrease c) neither
   d) neither

## I-11

1. a) bounded b) bounded below c) bounded
2. a) bounded above b) bounded above c) bounded
3. a) bounded b) bounded c) bounded below
4. a) bounded above b) bounded above c) not bounded
   d) bounded above

## I-12

1. a) continuous b) discontinuous c) continuous
2. a) continuous b) discontinuous c) discontinuous d) continuous
3. a) not continuous b) continuous c) continuous
4. a) continuous b) continuous c) continuous d) not continuous

## I-13

1. a) symmetric to x and y axes and origin
   b) symmetric to x and y axes and origin
   c) symmetric to x and y axes and origin
   d) symmetric to x axis

2. a) symmetric to origin
   b) symmetric to x axis
   c) symmetric to origin
   d) not symmetric to axes or origin

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### I-14

1. a) even b) odd c) odd d) even

2. a) neither b) odd c) odd d) odd

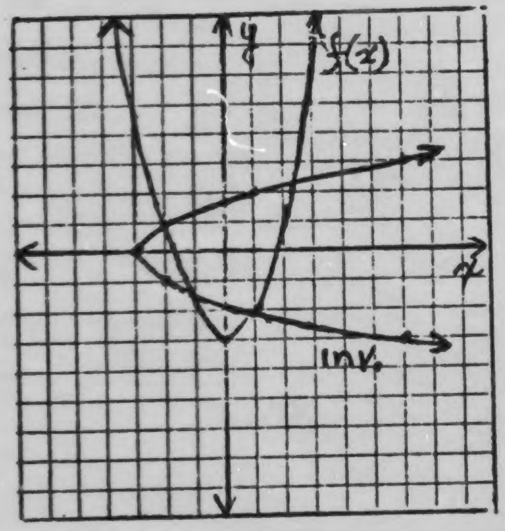
### I-15

1. a) $y = \pm\sqrt{x}$ b) $y = \sqrt[3]{x}$ c) $y = 3x + 6$ d) $y = \frac{1}{x}$

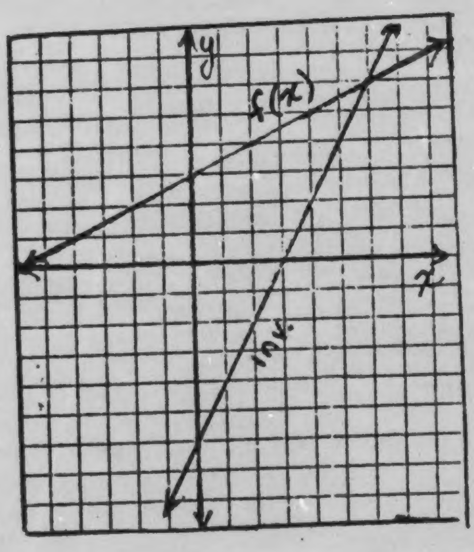
2. a) $y = \pm\sqrt{16 - x^2}$ b) $y = \frac{x}{x - 1}$ c) $y = \pm 2\sqrt{1 - x^2}$

### I-16

1.



2.



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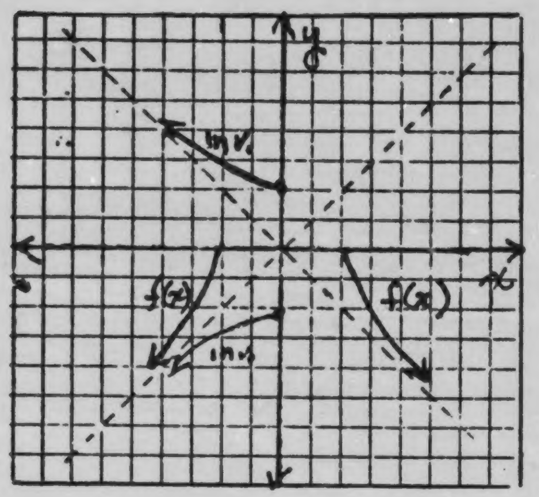
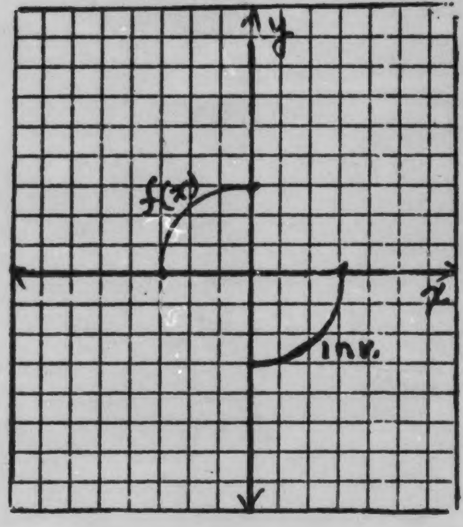
**3.**



**4.**



**I-17**

1. a) $D_1$ : $x \geq 0$ or $D_2$ : $x \leq 0$.

   b) $D_1$ : $x \geq 0$ or $D_2$ : $x \leq 0$.

   c) $D_1$ : $0 \leq x \leq 4$ or $D_2$ : $-4 \leq x \leq 0$

   d) $D_1$ : $x \geq 4$ or $D_2$ : $x \leq -4$

   e) $D_1$ : $x \geq 0$ or $D_2$ : $x \leq 0$.

   f) no modification required

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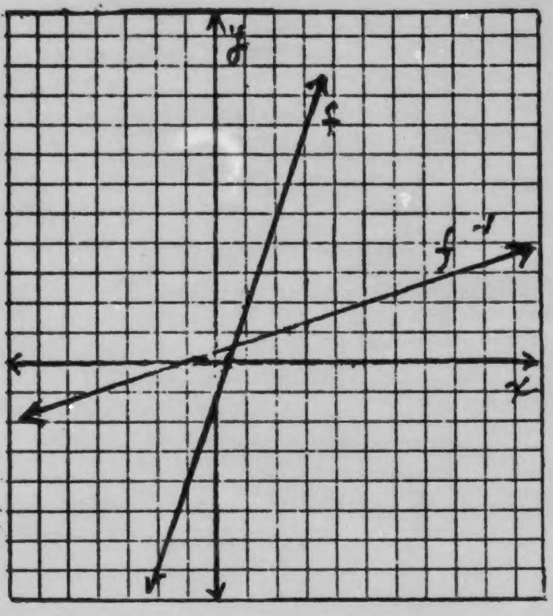
I-18

1. a)



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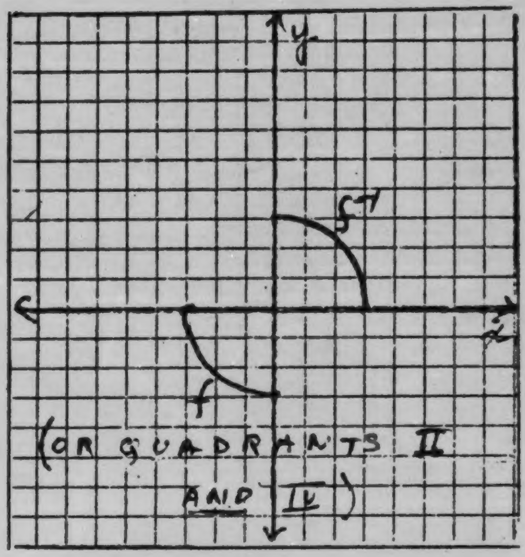
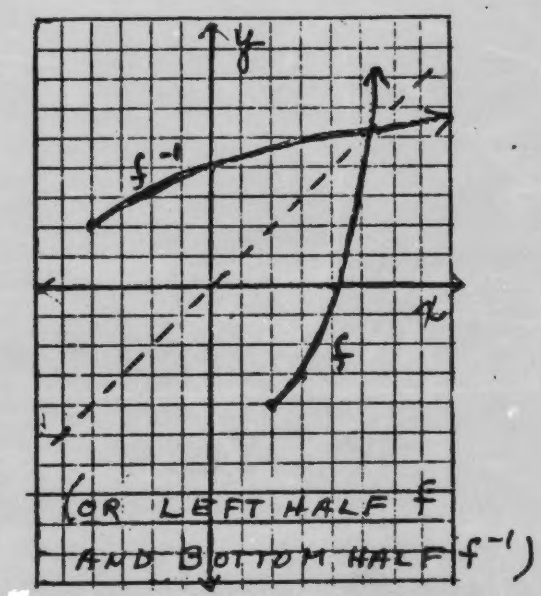


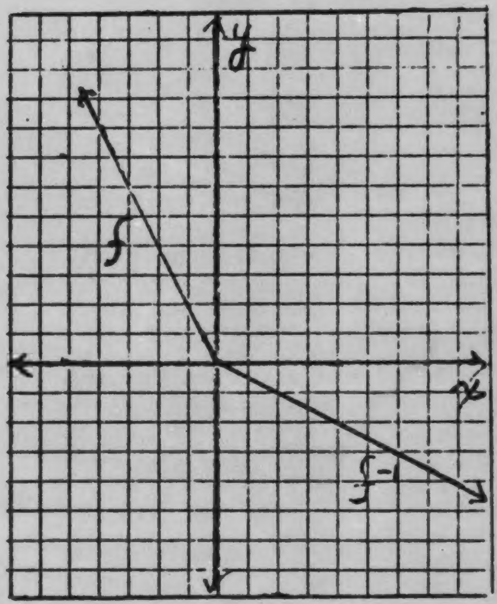
(OR QUADRANTS II AND IV)

c)



(OR LEFT HALF f AND BOTTOM HALF $f^{-1}$)

d)



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## I-19

1. Answers will vary, but suggested format follows:

   closure: $f(x) + g(x) = (f + g)(x)$

   associativity: $[f(x) + g(x)] + h(x) = f(x) + [g(x) + h(x)]$

   identity: $f(x) + I\,d(x) = f(x)$

   inverse: $f(x) + {}^{-}f(x) = I\,d(x)$

   commutativity: $f(x) + g(x) = g(x) + f(x)$

2. See #1 above.

3. Answers will vary. Suggested format:

$$f(x) \cdot [g(x) + h(x)] = f(x) \cdot g(x) + f(x) \cdot h(x)$$

## I-20

1. closure, associativity, identity, and inverse

2. Answers will vary.

3. $f(x) \circ f^{-1}(x) = f^{-1}(x) \circ f(x) = x = I\,d(x)$

   or

   $I\,d(x) \circ f(x) = f(x) \circ I\,d(x) = f(x)$

   (Examples will vary.)

Pre-Calculus - Elementary Functions

## Unit II Algebraic Functions

### Overview

Students should be familiar with polynomial functions and synthetic division from Algebra 2. In this unit these topics are extended, and rational and algebraic functions are added. An emphasis is placed on graphing techniques, including symmetry, intercepts, and the location of asymptotes. The ideas of the previous unit -- domain, range, increasing, decreasing, continuity, boundedness -- will be strengthened by additional study and application. Determination of roots and zeros will depend upon the application of the Intermediate Value Theorem. The field properties are fully satisfied when the study of polynomial functions is extended to include rational and algebraic functions.

### Suggestions to the Teacher

In order to complete the first seven objectives, the student is required to use several algebraic skills, including factoring and dividing polynomials, performing operations on complex numbers, and solving quadratic equations (the method of solving a quadratic is later extended in Objective 11 to factoring a fourth degree polynomial). The application of the Fundamental Theorem of Algebra will aid the student with Objectives 8 through 11. With Objectives 14 through 16, several methods for locating asymptotes may be studied. The discussion on asymptotes may introduce the idea of comparing the degree of the numerator to the degree of the denominator to determine the type of asymptotes (vertical, horizontal, or oblique). Solving the equation for x or considering the behavior of x as $x \to \pm\infty$ are two ways to locate horizontal asymptotes. The concept of a limit may be mentioned at this time. Symmetry to any point or line, and not just the origin (0,0) and the line x = 0, may be illustrated when graphing.

Because Objectives 11 and 16 are comprehensive, they may be used in place of one or more of the preceding objectives.

Some of the tedious work in the unit can be eliminated by using calculators and computers. A program for BASIC computer use, SOLPOL - Solving Polynomials, is available from Computer Related Instruction, in the Office of Instruction and Program Development.

Suggested time: 10 days

## Unit II Algebraic Functions

### PERFORMANCE OBJECTIVES

1. Identify polynomial functions when given a list of functions.

2. Determine the value of a polynomial function for given values of the variable using synthetic substitution and the Remainder Theorem.

3. Given two polynomials P and D where $D \neq 0$, determine the quotient and the remainder R, and express the result according to the Division Algorithm. [ $P(x) = Q(x) \cdot D(x) + R(x)$ ].

4. Apply the Factor Theorem to determine whether $x - c$ is a factor of a given polynomial function.

5. Apply the Factor Theorem to determine whether c is a zero of a given polynomial function.

6. Apply the Rational Root Theorem to determine the rational zeros of a polynomial function.

7. Apply the Intermediate Value Theorem and/or Locator Theorem to show the existence of zero(s) of a continuous function in a specified interval.

8. Graph a polynomial function.

9. Approximate the irrational zeros of a polynomial function.

10. Determine a polynomial function, given such information as the zeros and the degree.

**11. Given a polynomial function, determine all the zeros over the set of complex numbers and sketch the graph.

12. State the definition of a rational function.

13. State the domain of a rational function.

14. Determine the vertical and horizontal asymptotes of a given rational function.

15. Determine the oblique asymptotes of a given rational function. (optional)

**16. Sketch the graph of a rational function. Determine the domain, intercepts, and asymptotes. State whether the function is even, odd, or neither. Discuss symmetry.

17. Sketch the graph and determine the domain of a given algebraic function (non-polynomial and non-rational).

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**Comprehensive objectives

Unit II - Algebraic Functions

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 300-301 | 297-299 |  | 45-46 | 80 |  |
| 2 | 306-308 | 303-306 |  | 54-58 | 85-88 |  |
| 3 | 309-312 | 303, 306 |  | 56-58 | 81, 84 |  |
| 4 | 311-312 | 307-309 |  | 56-59 | 89, 91, 92 |  |
| 5 | 309-312 | 307-309 |  | 56-59 | 89-91 |  |
| 6 | 318-320 | 314-317 |  | 57-59 | 92-97 |  |
| 7 | 313-317 | 312-314 |  | 40-42 | 95-97 |  |
| 8 | 313-317<br>332-335 | 299-302<br>309-312 |  | 48-51<br>58-59 | 87-88<br>95-101 |  |
| 9 | 313-317 | 312-314<br>320-323 |  |  | 92-97 |  |
| 10 | 321, 324 | 312 |  | 58, 61 | 91-92 |  |
| 11 | 323-324<br>332-335 | 299-302<br>309-312<br>318-320 |  | 58 | 98-102 |  |
| 12 | 279, 341 | 323 |  | 62 | 120 |  |
| 13 | 280-283<br>341-344 |  |  | 62-64, 66 |  |  |
| 14 | 341-344 | 105-108<br>324-326 |  | 65-66 |  |  |
| 15 |  | 325-326 |  |  |  |  |
| 16 | 341-344 | 323-326 |  | 62-66 |  |  |
| 17 | 345-349 |  |  | 67-69 |  |  |

Pre-Calculus

Unit II

**II-1 Identify polynomial functions when given a list of functions.**

1. Write the word YES or NO to identify which of the following are polynomial functions.

_____ a) $y = x^2 - x + 5$

_____ b) $g(x) = \sqrt{x^2 - x}$

_____ c) $h(x) = \frac{1}{x-5}$

_____ d) $\{(x,y): y = 3^x\}$

2. Write the word YES or NO to identify which of the following are polynomial functions.

_____ a) $\{(x,y): y = 5\}$

_____ b) $f(x) = \sqrt{(x + 5)^5 (x - 1)}$

_____ c) $g(x) = 4 \sin 2x$

_____ d) $h(x) = x + \frac{x - 1}{x}$

3. Write the word YES or NO to identify which of the following are polynomial functions.

_____ a) $y = |x| - 1 + 5$

_____ b) $\{x \cdot y) : y = -2 \cos 3x\}$

_____ c) $g(x) = \frac{x^2 + 2x - 4}{5}$

_____ d) $h(x) = y - 10^{x-2}$

4. Write the word YES or NO to identify which of the following are polynomial functions.

_____ a) $f(x) = (2x - 1)^{-1}$

_____ b) $\{(x\ y): y = 4^x\}$

_____ c) $g(x) = (x - 1)^4 (2x - 5)$

_____ d) $h(x) = \sqrt[3]{x + 5}$

Pre-Calculus

Unit II

**II-2** Determine the value of a polynomial function for given values of the variable using synthetic substitution and the Remainder Theorem.

1. Use synthetic substitution and the Remainder Theorem to determine:

   $P(5)$ if $P(x) = 2x^4 - 5x^3 - x^2 - 10x - 15$.

2. Use synthetic substitution and the Remainder Theorem to determine:

   $P\left(-\frac{3}{2}\right)$ if $P(x) = x^4 + 3x^2 - 9$.

3. Use synthetic substitution and the Remainder Theorem to determine:

   $P(\sqrt{2})$ if $P(x) = 2x^3 - x^2 + x - 5$.

4. Use synthetic substitution and the Remainder Theorem to determine:

   $P(3 - i)$ if $P(x) = -x^4 + 2x^3 - 7x + 1$.

**II-3** Given two polynomials P and D where $D \neq 0$, determine the quotient and the remainder R, and express the result according to the Division Algorithm. $[P(x) = Q(x) \cdot D(x) + R(x)]$

1. Given $P(x) = x^3 - 2x^2 - x + 1$ and $D(x) = x - 3$, determine the $Q(x)$ and $R(x)$ and express the result in the form $P(x) = Q(x) \cdot D(x) + R(x)$.

2. Given $P(x) = 15x^3 + 16x^2 - 12x + 2$ and $D(x) = 3x + 5$, determine the $Q(x)$ and $R(x)$ and express the result in the form $P(x) = Q(x) \cdot D(x) + R(x)$.

3. Given $P(x) = \frac{1}{2}x^3 - \frac{1}{4}x^2 + x - 3$ and $D(x) = x^2 + 1$, determine the $Q(x)$ and $R(x)$ and express the result in the form $P(x) = Q(x) \cdot D(x) + R(x)$.

4. Given $P(x) = x^4 + 3x^3 - x^2 + 5x - 1$ and $D(x) = x + (2 - i)$, determine the $Q(x)$ and $R(x)$ and express the result in the form $P(x) - Q(x) \cdot D(x) + R(x)$.

**II-4 Apply the Factor Theorem to determine whether x - c is a factor of a given polynomial function.**

1. Use the Factor Theorem to determine which of the binomials $(3x + 2)$, $(x - 2)$ $(x - 4)$ are factors of $P(x) = 2x^3 + 5x^2 - 14x - 8$.

2. Use the Factor Theorem to determine which of the binomials $(x - 6)$, $(x - \sqrt{2})$, $(x - 3i)$ are factors of $P(x) = x^4 + 7x^2 - 18$.

3. Use the Factor Theorem to determine which of the binomials $(2x + 3)$, $(x + \frac{1}{2})$, $(x + 2)$ are factors of $P(x) = 2x^4 - 3x^3 - 23x^2 + 12x$.

4. Use the Factor Theorem to determine which of the binomials $(x + \sqrt{3})$, $(x - 5)$, $(x + 5i)$ are factors of $P(x) = x^3 + \sqrt{3}x^2 + 25x + 25\sqrt{3}$.

**II-5 Apply the Factor Theorem to determine whether c is a zero of a given polynomial function.**

1. Using the Factor Theorem, determine whether or not 3 is a zero of the polynomial function $P(x) = 2x^3 - 5x^2 - 2x - 3$.

2. Using the Factor Theorem, determine whether or not $-\frac{2}{3}$ is a zero of the polynomial function $P(x) = 6x^4 + x^3 - 5x^2 - 5x - 1$.

3. Using the Factor Theorem, determine whether or not $2 + \sqrt{3}$ is a zero of the polynomial function $P(x) = x^3 - 9x^2 + 21x - 5$.

4. Using the Factor Theorem, determine whether or not $-2 + 3i$ is a zero of the polynomial function $P(x) = x^3 + 5x^2 + 17x + 12$.

**II-6 Apply the Rational Root Theorem to determine the rational zeros of a polynomial function.**

Determine all the rational zeros of the following polynomials:

1. $P(x) = x^4 - 5x - 6$

2. $P(x) = 3x^3 - 14x^2 + 7x + 4$

3. $P(x) = 2x^3 - 11x^2 + 26x - 21$

4. $P(x) = x^3 - \frac{2}{3}x^3 + 3x - 2$

**II-7 Apply the Intermediate Value Theorem and/or Locator Theorem to show the existence of zero(s) of a continuous function in a specified interval.**

1. Given the continuous function $f(x) = x^2 - x - 3$, apply the Intermediate Value Theorem to show that there is a real zero in the interval $[-2, 0]$.

2. Given the continuous function $f(x) = x^3 + 2x^2 - 1$, apply the Intermediate Value Theorem to show that there is a real zero in the interval $[0, 1]$.

Pre-Calculus

Unit II

## II-8 Graph a polynomial function.

Graph the functions:

1. $P(x) = x(x + 2)(x - 1)$

2. $g(x) = (x - 3)^2 (x + 1)$

3. $A(x) = (x^2 + x - 6)(x^2 - 1)$

4. $P(x) = \frac{1}{2}x(x^3 - 8)$

## II-9 Approximate the irrational zeros of a polynomial function.

1. The polynomial function $P(x) = x^3 - 4x + 6$ has a zero between -3 and -2. Approximate this zero to the nearest tenth.

2. Approximate to the nearest tenth the smallest positive zero of $P(x) = x^3 + 6x^2 - 10x - 1$.

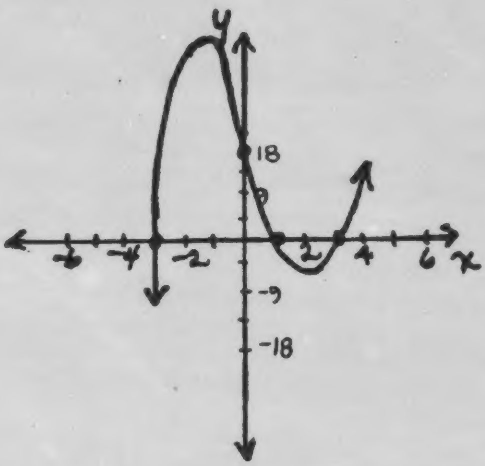
3. Approximate to the nearest tenth all real zeros of $P(x) = x^3 + 3x - 2$.

4. Given the polynomial function $P(x) = x^3 - x + 7$, approximate all real zeros of $P(x)$.

II-10 Determine a polynomial function, given such information as the zeros and the degree.

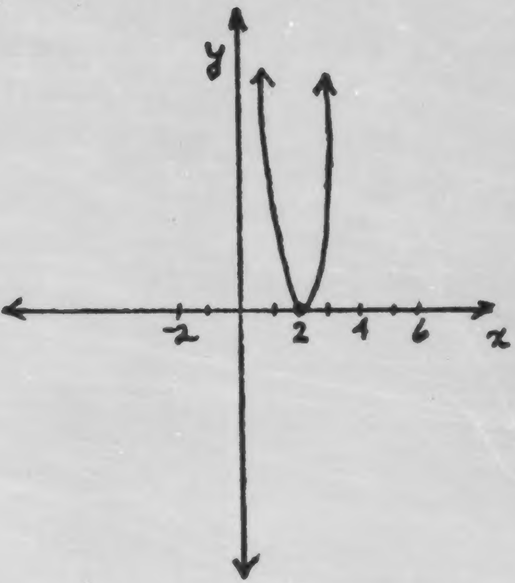
1. Determine the polynomial function of degree 3 having integral coefficients and zeros of $\frac{1}{2}$, 2, and $^{-}3$.

2. The graph of a polynomial function P(x) of degree 3 is shown at the right. Determine the polynomial function if its leading coefficient is 2.



3. Determine the third degree polynomial with leading coefficient of 1 and whose zeros are 2 + i, 3 - 2i, and i.

4. To the right, the graph of the polynomial function p(x) is of degree 6 and has leading coefficient of 4 and has only one distinct zero. Determine the polynomial function.



**II-11** Given a polynomial function, determine all the zeros over the set of complex numbers and sketch the graph.

Determine all zeros of the function and sketch its graph:

1. $p(x) = 2x^4 + x^3 - 25x^2 - 12x + 12$
2. $P(x) = x^4 - 9x^2 + 18$
3. $G(x) = 4x^4 + 14x^2 - 8$
4. $f(x) = x^3 + 3x^2 + 8x + 24$

**II-12** State the definition of a rational function.

1. Define a rational function in terms of two polynomial functions f and g.
2. State the definition of a rational function.
3. A rational function is ________________________________________

**II-13 State the domain of a rational function.**

1. State the domain of the function $r(x) = \frac{x}{x^2 + 1}$ over real numbers.
2. State the domain of the function $f(x) = \frac{4x}{9 - x^2}$ over real numbers.
3. State the domain of the function $g(x) \frac{3(x + 1)}{2x^2 + 5x + 3}$ over real numbers.
4. State the domain of the function $G(x) = \frac{12x^2 + 17x - 5}{2x^3 + 9x^2 - 5x - 42}$ over real numbers.

**II-14 Determine the vertical and horizontal asymptotes of a given rational function,**

Determine the vertical and/or horizontal asymptotes of the following rational functions:

1. $y = \frac{1}{2x - 4}$
2. $p(x) = \frac{2}{x^2 + 9}$
3. $p(x) = \frac{x^2 - 9}{x^2 - 3x - 10}$
4. $p(x) = \frac{3x^3 - x^2 + x + 5}{2x^3 - 5x^2 - 2x + 5}$

**II-15 Determine oblique asymptotes of a given rational function,**

1. Determine the oblique asymptote for the rational function $p(x) = \frac{x^2 + x + 1}{x - 1}$.

2. Oblique asymptote of the rational function $p(x) = \frac{x^2 - 2x + 2}{x + 2}$ is:

   a) $x = {}^{-}2$

   b) $y = x$

   c) $y = x - 4$

   d) $y = 1$

   e) There is no oblique asymptote.

3. Which one of the following functions has an oblique asymptote of $y = 2x$?

   a) $p(x) = \frac{2x^2 - 12x + 1}{x + 2}$

   b) $p(x) = \frac{2x^2 - 6x + 5}{x - 3}$

   c) $p(x) = \frac{x^2 + 3}{2x - 5}$

   d) $p(x) = \frac{2x^3 + 8x^2 + x - 5}{x^2 + 4}$

   e) $p(x) = \frac{(x^2 + 1)(x + 5)}{x^4 - 16}$

4. Determine the oblique asymptotes of $p(x) = \frac{(x - 2)(x + 4)(x - 3)}{(x - 1)(x + 3)}$.

**II-16** Sketch the graph of a rational function. Determine the domain, intercepts, and asymptotes. State whether the function is even, odd, or neither. Discuss symmetry.

1. Sketch the graph of $p(x) = \frac{3}{x - 4}$. Determine or discuss the indicated items regarding this function.

   a) domain

   b) coordinates of x and/or y intercepts

   c) symmetry with respect to y axis, origin

   d) asymptotes--vertical, horizontal, oblique

2. Given the function $p(x) = \frac{x}{x^2 - 9}$, sketch its graph. Determine or discuss the following:

   a) domain

   b) coordinates of x and/or y intercepts

   c) symmetry with respect to y axis, origin

   c) asymptotes

3. Determine the domain, coordinates of the x and y intercepts, symmetry, and asymptotes of the function, $G(x) = \frac{x^2 - 2x + 2}{x - 1}$. Sketch its graph.

4. Sketch the graph of $f(x) = \frac{8}{x^2 + 4}$, discussing domain, intercepts, symmetry, and asymptotes.

**II-17** Sketch the graph and determine the domain of a given algebraic function (non-polynomial and non-rational).

Determine the domain of the following functions and sketch a graph for each.

1. $y = \sqrt{x^2 - 9}$

2. $y = \frac{1}{\sqrt[3]{x}}$

3. $f(x) = 3 + \sqrt{x + 4}$

4. $g(x) = -\sqrt{\frac{(x - 2)}{x}}$

Pre-Calculus

Unit II

ANSWERS

## II-1

1. a) yes
   b) no
   c) no
   d) no

2. a) yes
   b) no
   c) no
   d) no

3. a) no
   b) no
   c) yes
   d) no

4. a) no
   b) no
   c) yes
   d) no

## II-2

1. $P(5) = 535$
2. $P(-3) = \frac{45}{16}$
3. $P(\sqrt{2}) = 5\sqrt{2} - 7$
4. $P(3 - i) = {}^-12 + 51\,i$

## II-3

1. $x^3 - 2x^2 - x + 1 = (x^2 + x + 2)(x - 3) + 7$

**Pre-Calculus**

**Unit II**

**ANSWERS**

II-3 (continued)

2. $15x^3 + 16x^2 - 12x + 2 = (5x^2 - 3x + 1)(3x + 5) - 3$

3. $\frac{1}{2}x^3 - \frac{1}{4}x^2 + x - 3 = (\frac{1}{2}x - \frac{1}{4})(x^2 + 1) + \frac{1}{2}x - \frac{11}{4}$

4. $x^4 + 3x^3 - x^2 + 5x - 1 = [\, x^3 + (1 + i)\, x^2 + (-4 - i)\, x + (14 - 2i]$

   $[\, x + (2 - i)\, ] + (-27 + 18\, i)$

II-4

1. $(x - 2)$

2. $(x + \sqrt{2}$ ; $(x - 3i)$

3. none

4. $(x + \sqrt{3})$; $(x + 5i)$

II-5

1. $P(3) = 0$; therefore 3 is a zero of $P(x)$.

2. $P(-\frac{2}{3}) = 1$; therefore $(-\frac{2}{3})$ is not a zero of $P(x)$.

3. $P(2 + \sqrt{3}) = 0$; therefore $(2 + \sqrt{3})$ is a zero of $P(x)$.

4. $P(-2 + 3i) = -1$; therefore $(-2 + 3i)$ is not a zero of $P(x)$.

II-6

1. $\{-1, 2\}$

2. $\{1, -\frac{1}{3}, 4\}$

3. $\{\frac{3}{2}\}$

4. $\{\frac{2}{3}\}$

Pre-Calculus

Unit II

ANSWERS

## II-7

1. $f(a) = f(-2) = +3$; $f(b) = f(-1) = -1$

   Let d be between f(a) and f(b) and be equal to 0. The theorem states that there exists a c such that $a < c < b$ and $f(c) = d = 0$. Thus c would be a zero of the function.

2. $f(a) = f(0) = -1$; $f(b) = f(1) = 2$.

   Let d be between f(a) and f(b) and be equal to 0. The theorem states that there exists a c such that $a < c < b$ and $f(c) = d = 0$. Thus c would be a zero of the function.

## II-8

See page 25.

## II-9

1. -2, 5
2. 1.4
3. .6
4. -2.1

## II-10

1. $p(x) = 2x^3 + x^2 - 13x + 6$
2. $p(x) = 2x^3 - 2x^2 - 18x + 18$
3. $p(x) = x^3 - 5x^2 + (9 + 4i)x + (-1 - 8i)$
4. $p(x) = 4(x - 2)^6$

## II-11

See page 26.

## II-12

Refer to textbook.

Pre-Calculus

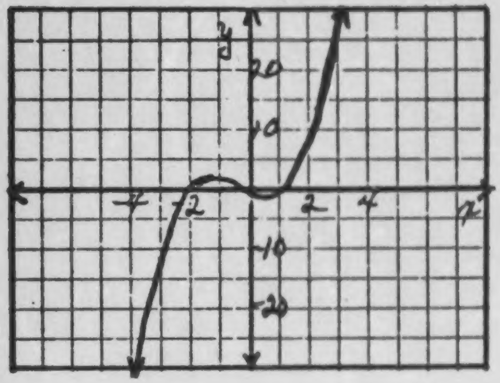
Unit II

ANSWERS

II-8

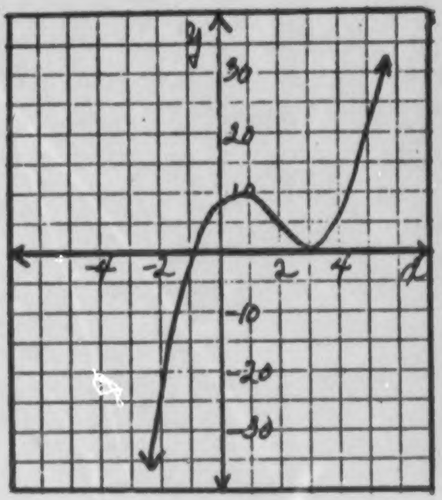
(1)

$P(x) = x(x+2)(x-1)$



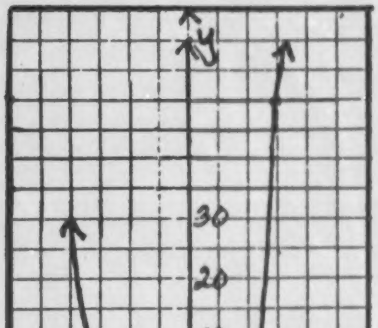
(2)

$g(x) = (x-3)^2(x+1)$



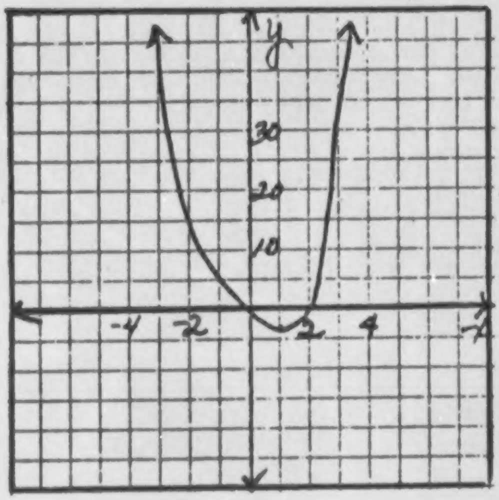
(3)

$A(x) = (x^2 + x - 6(x^2 - 1)$



(4)

$P(x) = \frac{1}{2} x (x^3 - 8)$



Pre-Calculus

Unit II

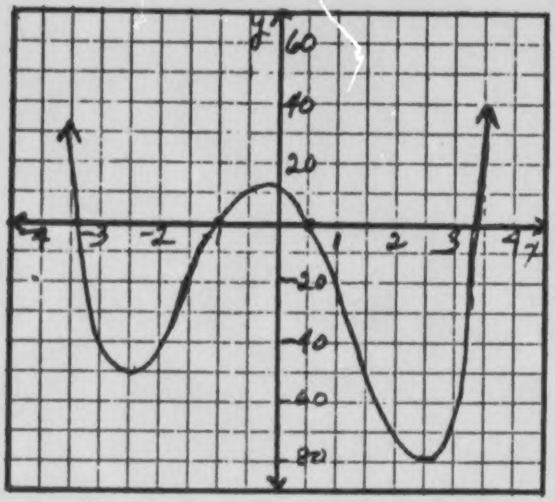
ANSWERS

II-11

(1)

$P(x) = 2x^4 + x^3 - 25x^2 - 12x + 12$

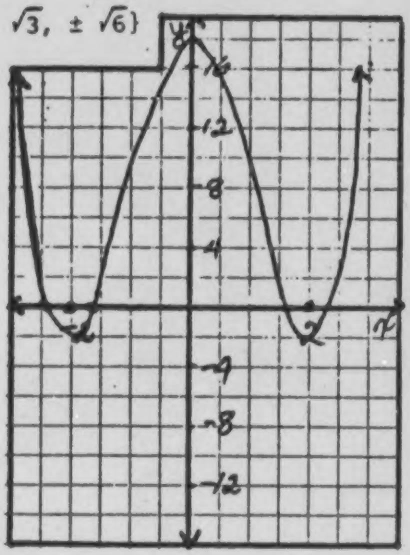
$\{\frac{1}{2}, -1, \pm\sqrt{12}\}$



(2)

$P(x) = x^4 - 9x^2 + 18$

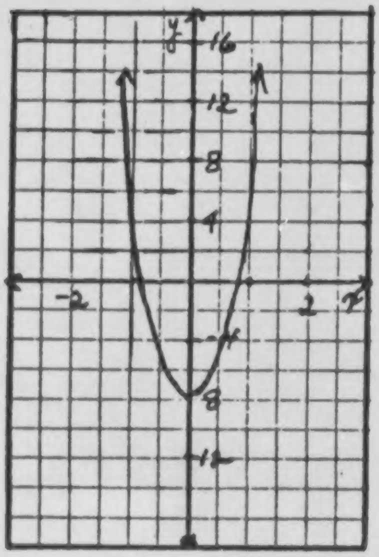
$\{\pm\sqrt{3}, \pm\sqrt{6}\}$



(3)

$G(x) = 4x^4 + 14x^2 - 8$

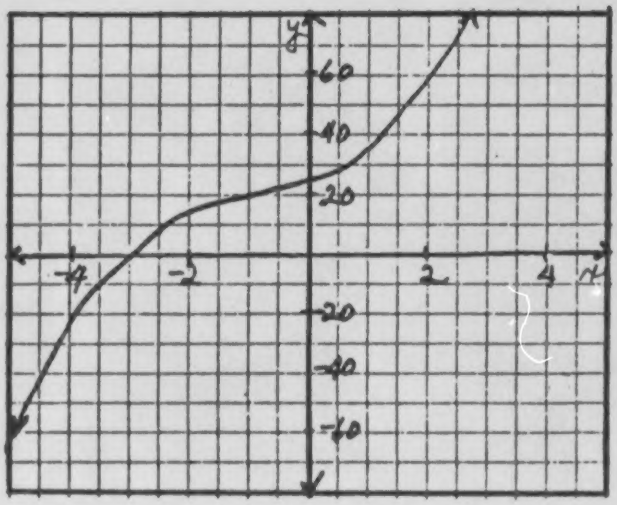
$\{\pm\frac{\sqrt{2}}{2}, \pm 2i\}$



(4)

$f(x) = x^3 + 3x^2 + 8x + 24$

$\{-3, \pm 2\sqrt{2i}\}$



Pre-Calculus

Unit II

ANSWERS

## II-13

1. all real numbers, R
2. $\{x: x \neq \pm 3\}$
3. $\{x: x \neq -1, -\frac{3}{2}\}$
4. $\{x: x \neq -3, -\frac{7}{2}, 2\}$

## II-14

1. $x = 2$; $y = 0$
2. $y = 0$
3. $x = 5$; $x = -2$; $y = 1$
4. $x = \frac{5}{2}$; $x = 1$; $x = -1$, $y = \frac{3}{2}$

## II-15

1. $y = x + 2$
2. c
3. b
4. $y = x - 3$

Pre-Calculus

Unit II

## ANSWERS

### II-16

1. $p(x) = \frac{3}{x - 4}$

   a) domain $\{x : x \neq 4\}$

   b) coordinates of

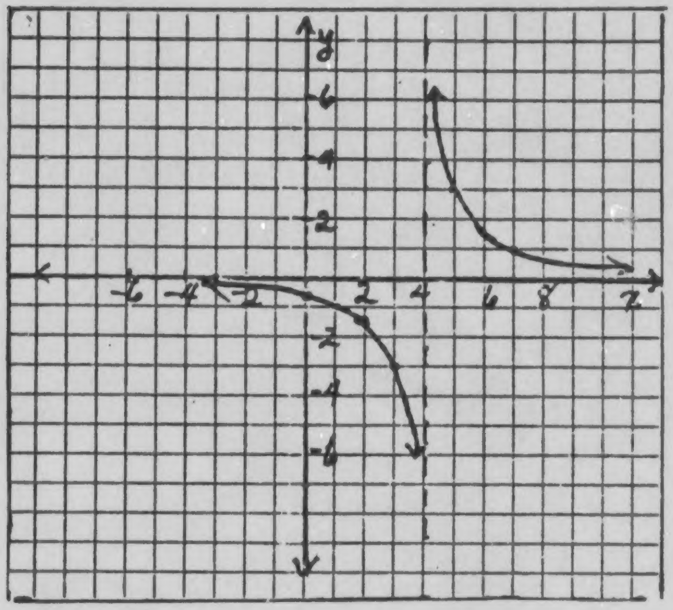
      x intercept(s): —
      y intercept: $(0, -\frac{3}{4})$

   c) symmetric with respect to

      y axis: no
      origin: no

   d) asymptotes

      vertical: $x = 4$
      horizontal $y = 0$
      oblique: none



2. $p(x) = \frac{x}{x^2 - 9}$

   a) domain $\{x : x \neq \pm 3\}$

   b) coordinates of

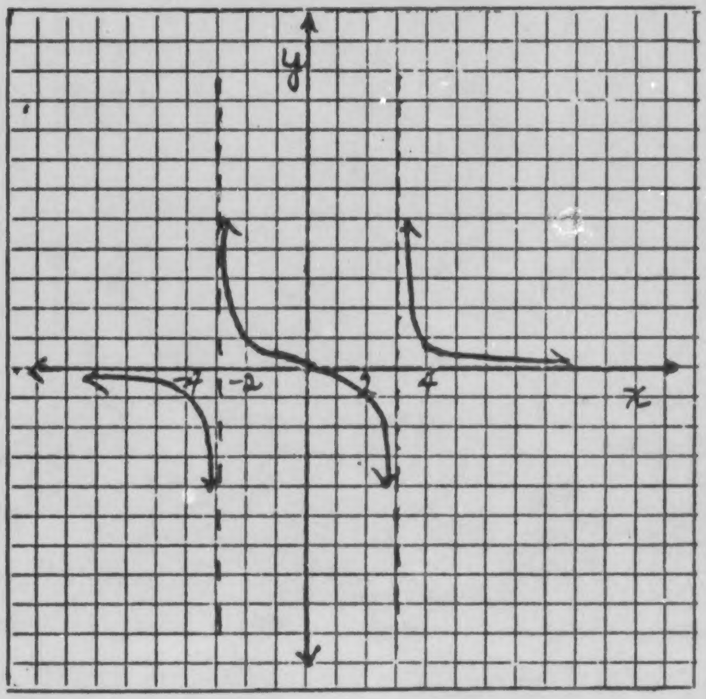
      $\left.\begin{array}{l}\text{x intercept}\\ \text{y intercept}\end{array}\right\} 0, 0$

   c) symmetric with respect to

      y axis: no
      origin: yes

   d) asymptotes

      vertical: $x = 3$, $x = -3$
      horizontal: $y = 0$
      oblique: none



Pre-Calculus

Unit II

ANSWERS

## II-16

3. $G(x) = \dfrac{x^2 - 2x + 2}{x - 1}$

   a) domain $\{x: \ x \neq 1\}$

   b) coordinates of

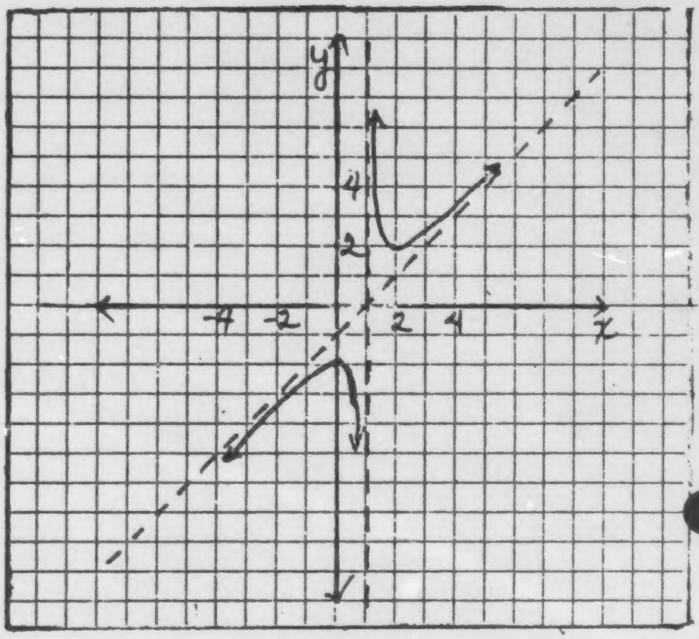
      x intercepts: —
      y intercept: (0, -2)

   c) symmetric with respect to

      y axis: no
      origin: no

   d) asymptotes

      vertical: $x = 1$
      horizontal: none
      oblique: $y = x - 1$



4. $f(x) = \dfrac{8}{x^2 + 4}$

   a) domain: $\{x: x \in R\}$

   b) coordinates of

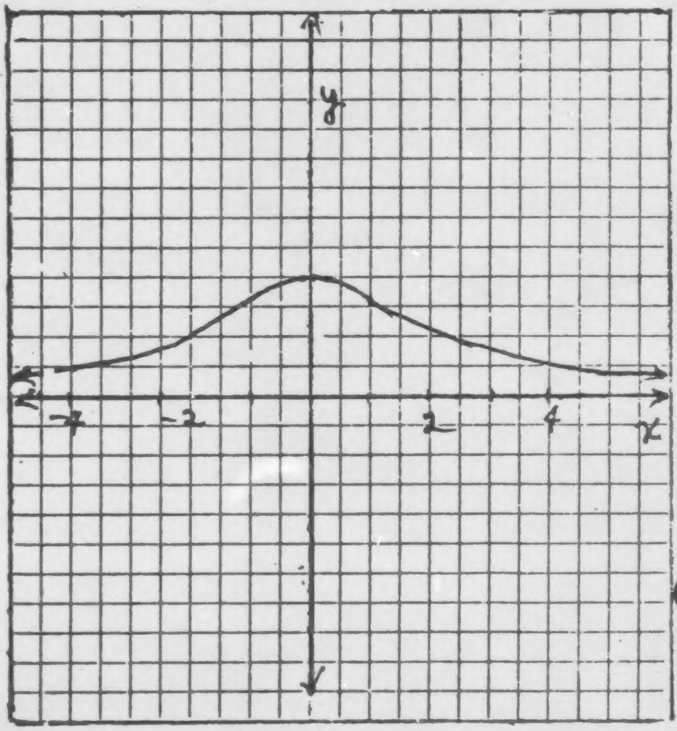
      x intercepts: —
      y intercept: (0, 2)

   c) symmetric with respect to

      y axis: yes
      origin: no

   d) asymptotes

      vertical: none
      horizontal: $y = 0$
      oblique: none



Pre-Calculus

Unit II

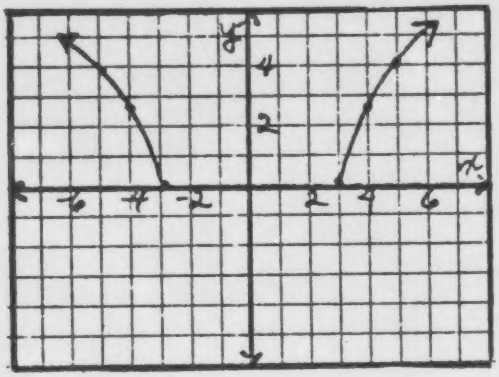
ANSWERS

II-17

(1)

$y = \sqrt{x^2 - 9}$

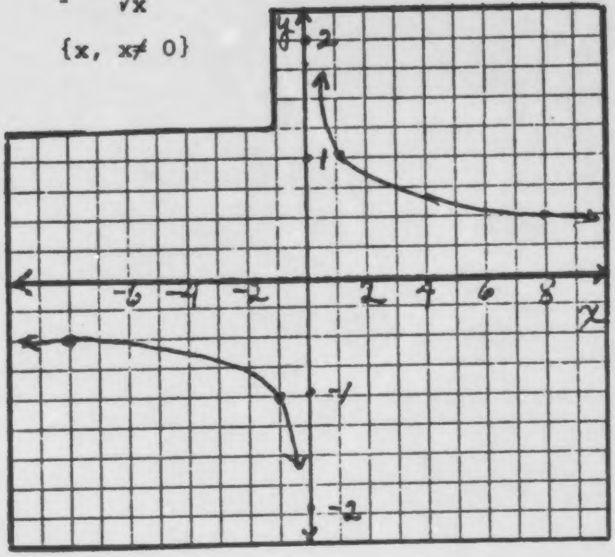
$\{x: x \geq 3 \text{ or } x \leq -3]$



(2)

$y = \dfrac{1}{\sqrt[3]{x}}$

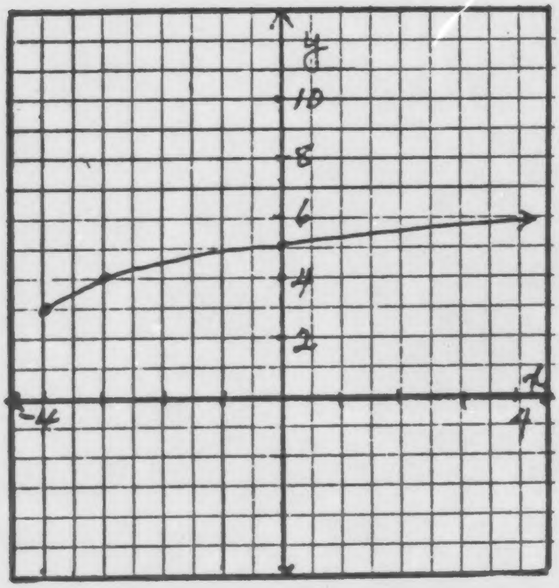
$\{x,\ x \neq 0\}$



(3)

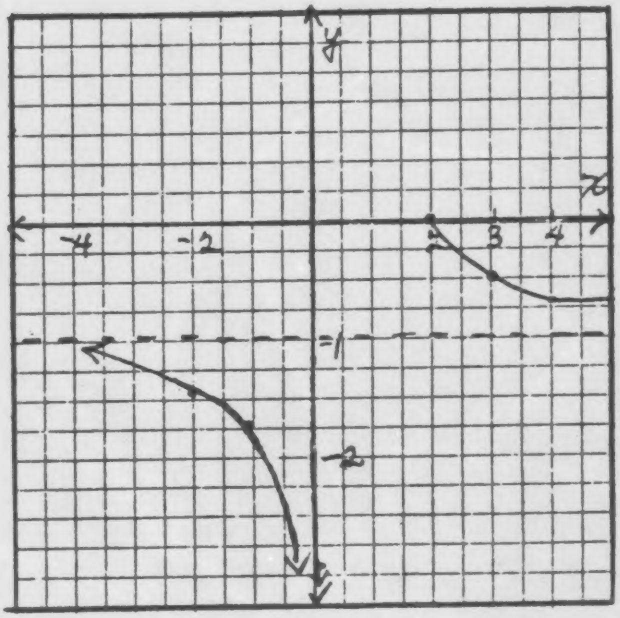
$f(x) = 3 - \sqrt{x + 4}$

$\{x: x \geq -4\}$



(4) $g(x) = \sqrt{\dfrac{x - 2}{x}}$

$\{x: x < 0 \text{ or } x \geq 2\}$



Pre-Calculus - Elementary Functions

## Unit III Exponential and Logarithmic Functions

### Overview

In this unit a systematic presentation of transcendental functions provides a comprehensive review and extension of real number exponents and logarithms. The study of the behavior of the functions, together with their graphs, is essential.

### Suggestions to the Teacher

A rigorous approach to the unit may be somewhat alleviated if the use of calculators is permitted. Emphasis should be placed on setting up an equation in order to solve a problem using logarithms. Proving the Laws of Logarithms is also appropriate at this level in the student's mathematical development. The constant $e$ and the function $\ln x$ should be included in the teaching of these objectives wherever possible. In addition, the amount of linear interpolation used in problems may be determined by class needs and background experiences.

Suggested time: 10 days

## Unit III Exponential and Logarithmic Functions

## PERFORMANCE OBJECTIVES

Note: The constant e and the function ln x should be included in the teaching of these objectives wherever possible.

1. Apply the Laws of Exponents to simplify expressions.
2. Sketch the graph of an exponential function.
3. Solve exponential equations using the Laws of Exponents.
4. Solve a growth or decay problem, using exponential functions.
5. Given an exponential function, discuss its properties (domain, range, continuity, increasing-decreasing, bounds, and one-to-one).
6. Sketch the graph of a logarithmic function.
7. Given a logarithmic function, discuss its properties (domain, range, continuity, increasing-decreasing, bounds, and one-to-one).
8. Solve logarithmic equations using the Laws of Logarithms.
9. Using the Laws of Logarithms, compute products, quotients, powers, or roots (or a combination of these operations).
10. Solve exponential or logarithmic equations, using a table of logarithms.
11. Use logarithms to solve verbal problems.

## Unit III - Exponents and Logarithms

### CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 362-365 | 334, 337 | | 73-76, 79 | 198-201 | |
| 2 | 366-372<br>376-379 | 335-338, 341 | | 78-80 | 235-239 | |
| 3 | | 350-351 | | 79 | | |
| 4 | 383-387 | 342-345 | | 81-82 | 252-257 | |
| 5 | 366-372<br>376-379 | 335-338<br>341-342 | | 80 | 235-239 | |
| 6 | 392-393<br>398 | 346-351 | | 85, 87-88 | 235-239 | |
| 7 | 392-393<br>397-399 | 346-351 | | 88 | 235-239 | |
| 8 | 402-403 | 347-350<br>355-357 | | 84, 89-90 | | |
| 9 | | | | | 208-213 | |
| 10 | 402-403 | 351<br>354-357, 362 | | 89-90 | | |
| 11 | 408-409 | 342-345 | | 90 | 252-257 | |

**III-1 Apply the Laws of Exponents to simplify expressions.**

Simplify. Write answers with positive exponents:

1.

a) $\dfrac{x^5\ y^{-3}}{x^6\ y^4}$

b) $\sqrt[3]{-8x^5y^9} \cdot \sqrt{4xy}$

c) $\dfrac{6x^4y^{-\frac{2}{3}}}{16^{\frac{1}{2}}x^{\frac{1}{3}}y^{\frac{1}{5}}}$

2.

a) $\dfrac{6x^2\ y^4}{18x^{-4}y^{\frac{1}{2}}}$

b) $\sqrt[3]{2a^7b^{-2}}\quad \sqrt[4]{8ab}$

c) $\dfrac{(9x^2y^3)^{\frac{1}{3}}}{3^{-1}x^{-4}y}$

3.

a) $\dfrac{-9y^{-2}}{27x^{-1}y^{-5}}$

b) $\sqrt{12x^{-2}y}\quad \sqrt[3]{9x^4y^8}$

c) $\dfrac{27^{\frac{2}{3}}a^4b^{-\frac{1}{2}}}{3a^{-\frac{2}{3}}b^{\frac{1}{2}}}$

4.

a) $\dfrac{-4x^{-2}y^{-4}}{64x^{-1}\ y^5}$

b) $\sqrt[4]{2xy}\quad \sqrt[6]{16x^{-1}y^2}$

c) $\dfrac{64^{\frac{2}{3}}\ a^{\frac{1}{2}}\ b}{32^{\frac{1}{5}}\ a^{-\frac{3}{4}}b^{-\frac{2}{5}}}$

## III-2 Sketch the graph of an exponential function.

Sketch the graph of:

1. $E(x) = 2^{x - 1}$
2. $f(x) = 2^{-x} + 3$
3. $g(x) = 3^{-x^2}$
4. $H(x) = 3^{-x} + 3^{x + 2}$

## III-3 Solve exponential equations using the Laws of Exponents.

Solve for the variable:

1. x: $2^{-x} = 64^{\frac{2}{3}}$
2. y: $\frac{1}{2^{y-1}} = 32^{\frac{3}{5}}$
3. a: $(a + 2)^{\frac{1}{6}} = 3^{\frac{2}{3}}$
4. y: $9\ {}^{8}y + 3 = 27^{y^2 -2}$

**III-4 Solve a growth or decay problem, using exponential functions.**

1. The number N of fruit flies at time t days is given by the equation $N = 100 \cdot 2^{t/4}$. How many flies are present at the beginning of the experiment? How long does it take to double the population of fruit flies?

2. If a new $6000 car depreciates 20% each year, what will its value be in t years?

3. A bacteria population triples every 6 days. If the population is now 90, when will the population be 810?

4. What amount of money is reached by investing $400 for 5 years at 6% interest compounded continuously?

III-5 Given an exponential function, discuss its properties (domain, range, continuity, increasing-decreasing, bounds, and one-to-one)

Discuss the properties of the function (domain, range, continuity, increasing, decreasing, bounds, and one-to-one) for:

1. $E(x) = 2^{-x}$
2. $f(x) = 3^{\frac{1}{x}}$
3. $G(x) = 2^{x+3} - 1$
4. $p(x) = e^{2x}$

III-6 Sketch the graph of a logarithmic function.

Sketch the graph of the following functions:

1. $f(x) = \log_2(x - 1)$
2. $g(x) = \log_3 \sqrt{x}$
3. $f(x) = \log_{\frac{1}{3}}(-x)$
4. $h(x) = \ln|x|$

III-7 Given a logarithmic function, discuss its properties (domain, range, continuity, increasing-decreasing, bounds, and one-to-one).

Discuss the properties of the function (domain, range, continuity, increasing - decreasing, bounds, and one-to-one):

1. $f(x) = \log_{\frac{1}{2}} x$
2. $g(x) = \log_3 (-2x)$
3. $h(x) = (\log_3 x)^2$
4. $G(x) = |\ln x|$

III-8 Solve logarithmic equations using the Laws of Logarithms.

Solve for x:

1. $\log_3 x + \log_3 (x + 2) = 1$
2. $\log_2 (2x + 3) - \log_2 (x - 3) = \log_2 5$
3. $\log 4 - \log (x^2 - 9) + \log (x + 3) = \log x$
4. $\log x = \frac{1}{3}[2 \log 8 - 6 \log 3] - 2 \log 2 + \log 3$

III-9 Using the Laws of Logarithms, compute products, quotients, powers, or roots (or a combination of these operations).

Compute the product, using logarithms:

1. $(6.52)\ (31.4)$

2. $\frac{.01484}{3.015}$

3. $(.0157)^{\frac{1}{3}} \cdot (11.99)^{2}$

4. $\frac{-(738600)(.2743)^{2}}{\sqrt{(8.1)\ (48.12)}}$

III-10 Solve exponential or logarithmic equations, using a table of logarithms.

Using a table of logarithms, solve for each variable:

1. $3^{x} = 1208$

2. $N = \log_{5} 28.14$

3. $3^{2x+1} = 2^{x-4}$

4. $19.4^{x-1} = 82.18$

## III-11 Use logarithms to solve verbal problems.

1. If \$425 is invested at 6% compounded quarterly, how much money will accumulate in 12 years?

2. A house bought 2 years ago for \$35,560 was recently sold for \$45,900. Assuming the value of the house increases exponentially, what will the value of the house be in two more years? (to the nearest dollar)

3. The equation for the amount of energy E exerted by an object moving f feet per second and weighing w pounds is $E = \frac{wf^2}{2g}$ where E is measured in foot pounds and g is the force of gravity. What is the weight of the automobile (to the nearest hundred) if it strikes an object while traveling 55 miles per hour and exerts 354,087 foot pounds of energy? (Use $g = 32.16$.)

4. The period P of a simple pendulum is given by $P = 2\pi\sqrt{\frac{L}{g}}$ where P is in seconds, L is the length of the pendulum arm, and g is about 10 meters per-second-per-second. If the period of a pendulum is 3.264 seconds, what is the length of the pendulum arm? (Use $\pi = 3.14$.)

**Pre-Calculus**

**Unit III**

**ANSWERS**

## III-1

1. a) $\dfrac{1}{xy^7}$

   b) $-4x^{\frac{13}{6}}\ y^{\frac{7}{2}}$ or $-4x^2y^3\sqrt[6]{xy^3}$

   c) $\dfrac{3x^{\frac{11}{3}}}{2y}$

2. a) $\dfrac{x^6y^{\frac{7}{2}}}{3}$

   b) $\dfrac{2^{\frac{13}{12}}\ a^{\frac{31}{12}}}{b^{\frac{5}{12}}}$ or $\dfrac{2a^2\ \sqrt[12]{2a^7b^7}}{b}$

   c) $3^{\frac{5}{3}}\ x^{\frac{14}{3}}$ or $3x^4\sqrt[3]{9x^2}$

3. a) $-\ \dfrac{xy^3}{3}$

   b) $2 \cdot 3^{\frac{7}{6}}\ x^{\frac{1}{3}}\ y^{\frac{19}{6}}$ or $6y^3\sqrt[6]{3x^2y}$

   c) $\dfrac{3a^{\frac{14}{3}}}{b}$

4. a) $-\ \dfrac{1}{16xy^9}$

   b) $2^{\frac{11}{12}}\ x^{\frac{1}{12}}\ y^{\frac{7}{12}}$ or $\sqrt[12]{2^{11}x\ y^7}$

   c) $8a^{\frac{5}{4}}\ b^{\frac{7}{5}}$ or $8\ \sqrt[20]{a^5b^8}$

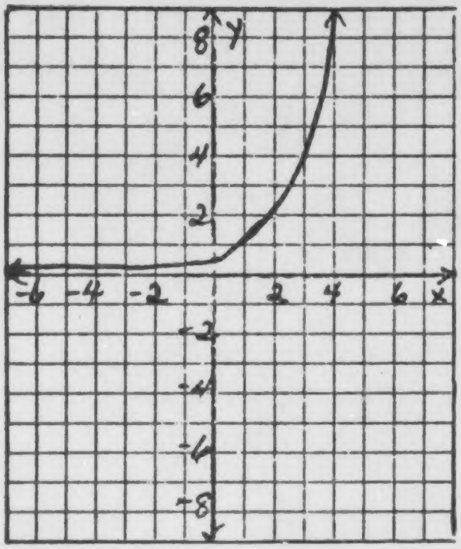
Pre-Calculus

Unit III

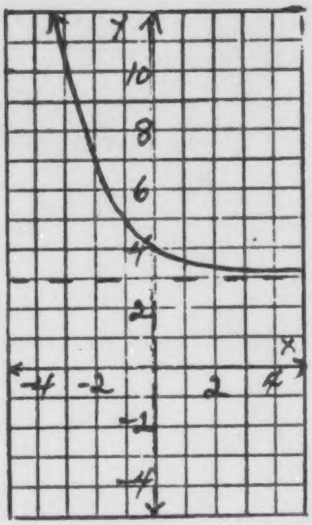
ANSWERS

## III-2

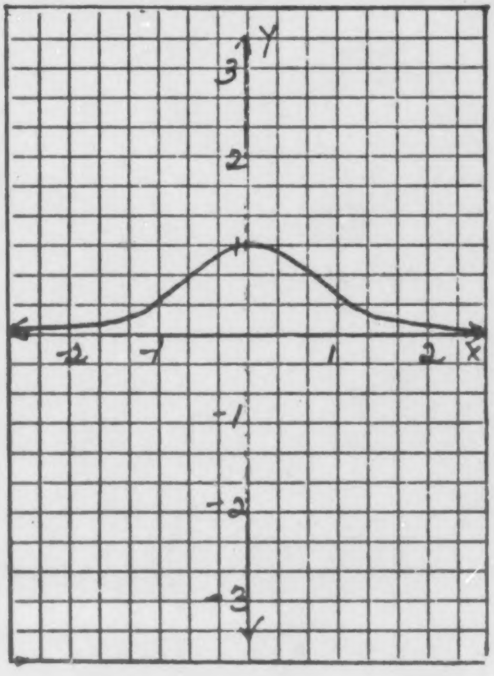
a)



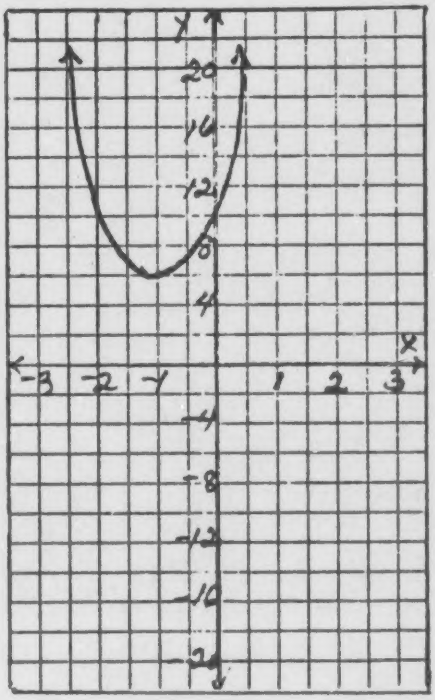
b)



c)



d)



Pre-Calculus

Unit III

ANSWERS

## III-3

1. $\{-4\}$
2. $\{-2\}$
3. $\{79\}$
4. $\{-\frac{2}{3}, 6\}$

## III-4

1. 100, 4 days
2. $C = 6000\ (.8)^t$
3. 12 days
4. $539.96

## III-5

1. domain: all real numbers

   range: all real numbers >0

   continuous function

   decreasing, one-to-one

   Greatest lower bound is 0.
2. domain: all real numbers except 0

   range: all real numbers >0

   continuous everywhere except at $x = 0$

   decreasing, one-to-one

   Greatest lower bound is 0.

Pre-Calculus

Unit III

ANSWERS

III-5 (continued)

3. domain: all real numbers

   range: all real numbers > - 1

   continuous function

   increasing, one-to-one

   Greatest lower bound is -1.

4. domain: all real numbers

   range: all real numbers > 0

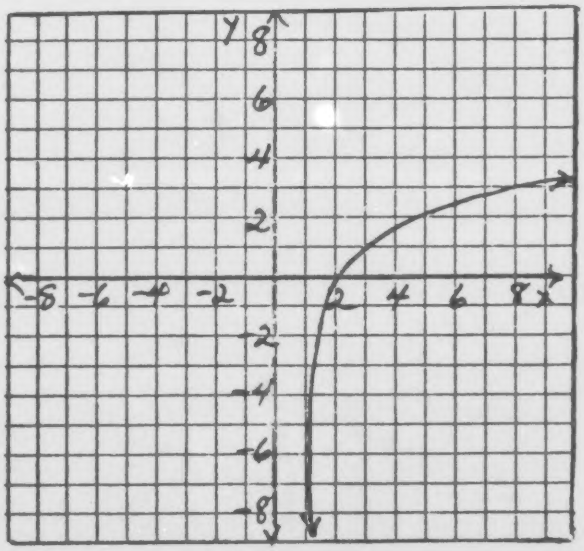
   continuous function

   increasing, one-to-one

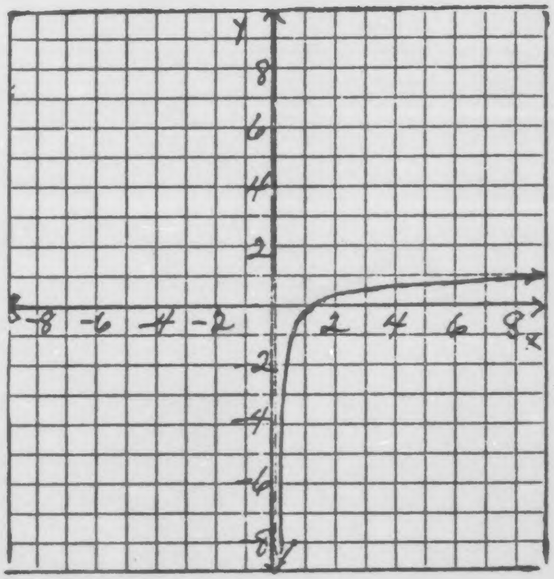
   Greatest lower bound is 0.

III-6

1.

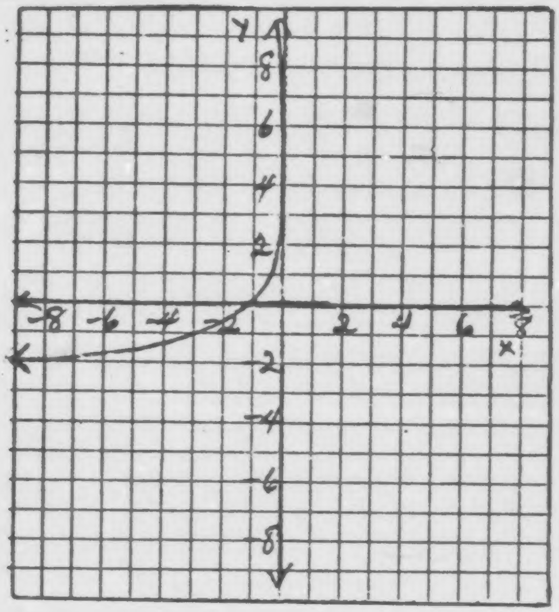


2.

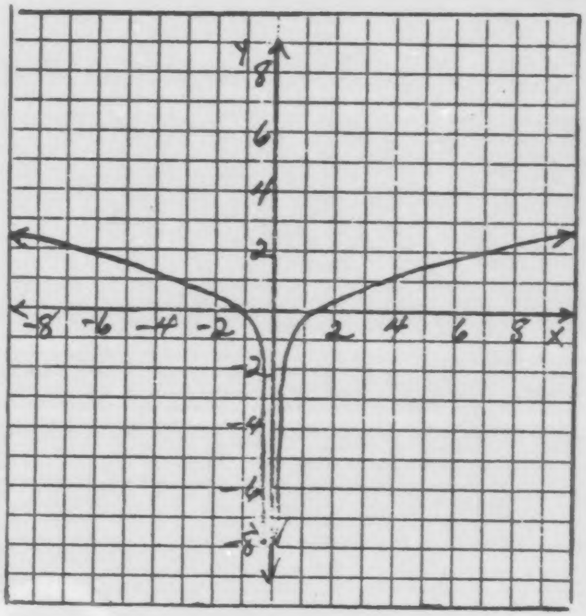


III-6 (continued)

3.



4.



III-7

1. domain: all real numbers > 0

   range: all real numbers

   continuous function

   decreasing, one-to-one

2. domain: all real numbers < 0

   range: all real numbers

   continuous everywhere except at x = 0

   decreasing, one-to-one

### III-7 (continued)

3. domain: all real numbers $> 0$

   range: all real numbers $\geq 0$

   continuous function

   increasing, one-to-one

4. domain: all real numbers $> 0$

   range: all real numbers $\geq 0$

   continuous function, one-to-one

   decreasing for $0 < x < 1$ and increasing for $x \geq 1$

   Greatest lower bound is 0.

### III-8

1. $x = 1$
2. $x = 6$
3. $x = 4$
4. $x = \frac{1}{3}$

### III-9

1. 204.7
2. .004921
3. 36
4. -2815

Pre-Calculus

Unit III

ANSWERS

## III-10

1. 6.46
2. 2.074
3. -2.574
4. 2.487

## III-11

1. $868.50
2. $59,247
3. 3500 pounds
4. 2.701 meters

**Pre-Calculus** - Elementary Functions

## Unit IV Circular Functions

### Overview

Each student should have received an introduction to circular functions prior to this unit. The purpose of this phase of instruction is to increase the student's knowledge of circular functions.

It is possible that periodic functions were discussed in Unit I; however, circular functions are the first specific examples of periodic functions which the student has encountered.

### Suggestions to the Teacher

Applications of the property of periodicity should be stressed as well as relating and reinforcing the properties of functions (increasing, decreasing, continuity, boundedness, one-to-one, etc.) which were studied in Units I, II, and III.

Evaluating a function, sketching the graphs of circular functions of the form $y - \cos^{*} (b\theta + c) + d$, and determining the equation of the function are skills which should be handled with ease by the student.

Objectives 1-14, except 7 and 13, should be covered rapidly, as these objectives have been taught in previous courses. Radian measurement of an angle is implied unless the ° symbol is used. Determining the value of equations such as $\cos 1.4567 = x$ or $\cos \theta = .4560$ should be emphasized.

When sketching the graphs of the six circular functions (Objective 6), include variations of the basic functions, such as $y = \cos \theta + 4$, $y = \cos \theta$; $y = -\cos \theta$, etc.

The instruction of Objective 9 should include evaluating expressions such as $(\sin \frac{\pi}{3} + \cos \frac{\pi}{4})^2$. The student should maintain good algebraic skills and the functional value of these multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ should be known by the student without the use of a notecard.

Objective 7 is comprehensive in nature. The student should be able to relate the various properties of functions in general to a specific set of functions, e.g., the circular functions.

Many objectives in this unit may be taught together (e.g., Objectives 9-10; 11-12; and 18-22). If Objective 22 is taught, it will be necessary for the student to acquire skills in solving equations of the form $\cos [\frac{\pi}{6}(t - 2)] = .4567$. Additional problems of this type are indicated in sample problem #4 for Objective 12. In evaluating radian measurement, the following approximations were used: $\frac{\pi}{2} \doteq 1.5708$; $\pi \doteq 3.1416$; $\frac{3}{2}\pi \doteq 4.7124$, and $2\pi \doteq 6.2832$. Because of these approximations, answers may vary for some of the sample problems.

---

*any circular function

Suggestions to the Teacher (continued)

Stress Objectives 14-16. The concept of periodicity is again involved. The period can be expressed in units of π or simply in numbers such as a period of 10. In the sample problems for Objective 16, answers may again vary depending on the phase shift selected by the student.

More can be done with the formula for f + g (Objective 17) after discussing the sum and difference formulas in Unit V.

Objectives 18-22 allow the student to see practical and useful applications of circular functions. The best textbook for these objectives is Trigonometry: Functions and Applications, by Foerster.

It is recommended that each school have copies of these textbooks:

a. Trigonometry: Functions and Applications, by Foerster

b. Pre-Calculus Mathematics, by Crosswhite, Hawkinson, and Sachs

c. Pre-Calculus Mathematics, by Shank

The use of calculators is important in the solution of some of the application problems.

Suggested Time: 12 days

Pre-Calculus

## Unit IV Circular Functions

### PERFORMANCE OBJECTIVES

1. Convert angle measurement from degrees to radians or radians to degrees.

2. Solve verbal problems using the formula for arc length, $s = r\theta$.

3. Use the definition of the wrapping function to determine functional values for a specific value of the domain.

4. State the definitions of the six circular functions.

5. State the domain and range for each of the circular functions.

6. Sketch the graphs of the six circular functions.

** 7. Given a circular function, indicate its properties with respect to continuity, asymptotes, increasing-decreasing, even-odd, boundedness, and periodicity.

8. Given an angle, determine its reference angle.

9. Determine the functional values of the special angles (multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$).

10. Determine the measure(s) of a special angle, given the functional value of the angle.

11. Use a table to determine the functional value of a given angle, interpolating as necessary.

12. Use a table to determine the measure(s) of an angle, given the functional value of an angle, interpolating as necessary.

** 13. Given the equation or graph of a circular function, determine the period, amplitude, phase shift, and vertical shift.

14. Sketch the graph of a circular function that has a phase shift and/or vertical shift.

15. Given the amplitude, period, phase shift, and vertical shift, write the equation of a sine or cosine function.

16. Given the graph of a circular function, determine its equation.

17. Given two circular functions, f and g, sketch the graph of f + g by addition of ordinates (graphical addition).

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**comprehensive objectives

## Unit IV Circular Functions

PERFORMANCE OBJECTIVES (continued)

18. Solve verbal problems involving uniform circular motion.

19. Solve verbal problems involving a simple harmonic motion.

### Optional

20. Solve verbal problems involving other applications of the circular functions (alternating current, radio, Fourier coefficients).

21. Construct the equation of a sinusoidal function given data from a set of physical science observations.

22. Use the equations constructed in Objective 21 to make predictions.

Unit IV - Circular Functions

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 115-118 | 165-169 | | 97-100 | 276-280 | |
| 2 | 119 | | | 98-100 | 281 | |
| 3 | 65-68 | 125-127 | | 100-105 | 265-270 | |
| 4 | 70, 97 | 129-132 | | 105, 108 | 270, 313 | |
| 5 | 70, 97<br>124-126 | 132-134<br>139-140 | | 114-115 | 270, 313 | |
| 6 | 82-83<br>98-102 | 132-141 | | 114-115 | 281-287<br>313-316 | |
| 7 | 102 | | | 115-116 | 272-273 | |
| 8 | 120 | | | 129-133 | | |
| 9 | 74-81<br>127 | 130-132 | | 111-113 | 271-272 | |
| 10 | 74-81 | 130-132 | | 111-113 | 271-276 | |
| 11 | | 170-172 | | 134-135 | | |
| 12 | | 170-172 | | 134-135 | | |
| 13 | 86-91<br>92-96 | 126, 132-141<br>161 | | 114-121 | 281-287<br>293-298 | |
| 14 | 95-96 | 132-141<br>161 | | 114-121 | 281-287<br>293-298 | |
| 15 | 92-95 | | | | 286 | |
| 16 | 90-91 | 138 | | 118 | | |

Unit IV - Circular Functions (continued)

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 17 |  | 137-138 |  | 122-124 | 296-298 |  |
| 18 |  | 184-186 |  |  | 327-329<br>331-334 |  |
| 19 |  |  |  | 143-146 | 330-334 |  |
| 20 |  |  |  | 147-152 |  |  |
| 21 | For Objectives 21 and 22, refer to Trigonometry: Functions and Applications, Foerster, pp. 54-93. |  |  |  |  |  |
| 22 |  |  |  |  |  |  |

Pre-Calculus

Unit IV

IV-1 Convert angle measurement from degrees to radians or radians to degrees.

1. Convert each of the following angles into its equivalent degree or radian measurement.

| | Degree | Radian |
|---|---|---|
| a) | 35° | ____ |
| b) | -742° | ____ |
| c) | ____ | $\frac{7}{12}\pi$ |
| d) | ____ | 2.7 |

2. Change each angle to its equivalent degree or radian measurement.

| | | |
|---|---|---|
| a) | 205° | ____ |
| b) | 12° 15' | ____ |
| c) | ____ | $\frac{9}{64}\pi$ |
| d) | ____ | 1.6 |

3. Determine the equivalent degree or radian measurement of each of the following angles.

| | | |
|---|---|---|
| a) | π° | ____ |
| b) | ____ | π |
| c) | ____ | .5 |
| d) | 45° 6' 15" | ____ |

IV-2 Solve verbal problems using the formula for arc length, $s = r\theta$.

1. If the radius of a bicycle wheel is 28 inches, find the distance the wheel moves as the wheel travels through an angle of 10 radians.

2. True or False:

   A 30° arc in a circle of radius 40 is 4 times as long as a 30° arc in a circle of radius 10.

3. Friction gears use friction to transmit motion from one gear to another. If a friction gear with redius of 12 inches moves through an angle of 7 radians, determine through how many radians the contact gear with radian of 9 inches moves.

4. A bucket is drawn from a well by pulling the rope over a pulley. Find the radius of the pulley if the bucket is moved 82.9 inches while the pulley is turned through 4.4 revolutions.

IV-3 Use the definition of the wrapping function to determine functional values for a specific value of the domain.

1. Use the definition of the wrapping function to determine $W(\frac{11}{2}\pi)$.

2. If $W(\theta)$ represents the wrapping function, determine $W(\frac{7}{6}\pi)$.

3. Determine $W(-\frac{7}{4}\pi)$ where W represents the wrapping function.

4. Using the definition of the wrapping function, determine $W(-\pi + 2\pi K)$, $K \in$ integers.

**IV-4 State the definitions of the six circular functions.**

1. State the definition of sin x.

2. The cos x is defined as __________ .

3. If W($\theta$) = (x, y) corresponds to the point on the unit circle associated with the real number $\theta$ by the wrapping function W, then

   sin $\theta$ = __________

   cos $\theta$ = __________

   tan $\theta$ = __________

4. State the definition of csc x in terms of the wrapping function.

IV-5 State the domain and range for each of the circular functions.

1. State the domain and range of the tangent function.

2. The domain and range of the sine function is __________.

3. The domain of the cos x is __________ while the range is __________.

4. State the domain and range of the cosecant function.

**IV-6 Sketch the graphs of the six circular functions.**

1. Sketch the graph of $y = \sin x$ for all $0 \le x \le 2\pi$.

2. Graph $y = \tan x$ for $-\frac{3}{2}\pi \le x \le \frac{3}{2}\pi$.

3. For all x, sketch the graph of $y = \cos x$.

4. Sketch the graph of $y = \sec x$ for $-2\pi \le x \le 2\pi$.

IV-7 Given a circular function, indicate its properties with respect to continuity, asymptotes, increasing-decreasing, even-odd, boundedness, and periodicity.

1. Given the graph of $y = \sin x$ as shown, discuss the following properties with respect to this graph:

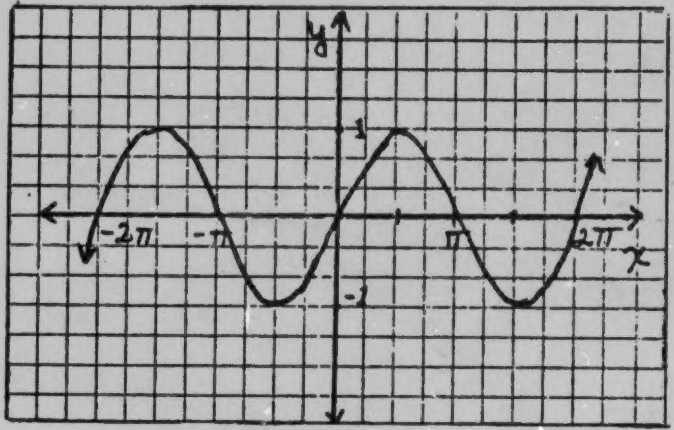
   a) continuity

   b) symmetry with respect to?

   c) boundedness

   d) period

   e) Indicate an interval for which $f(x)$ is an increasing function over $-\pi \leq x \leq 0$.



2. Given the graph of the tangent function as shown, indicate the properties of this function with respect to:

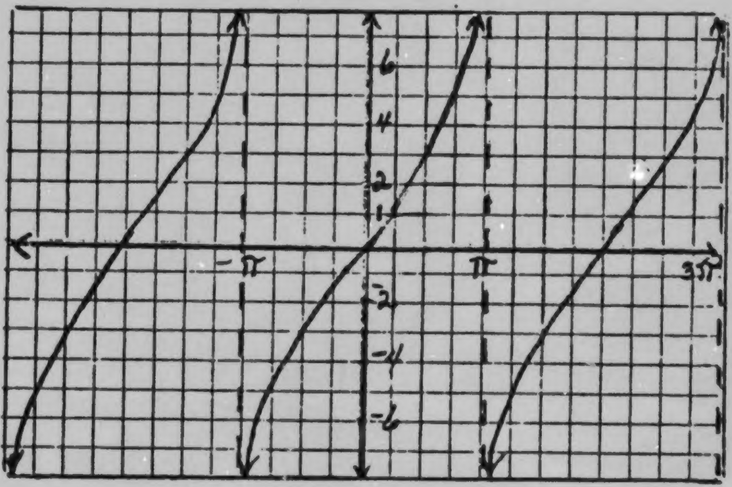
   a) even or odd function

   b) increasing or decreasing

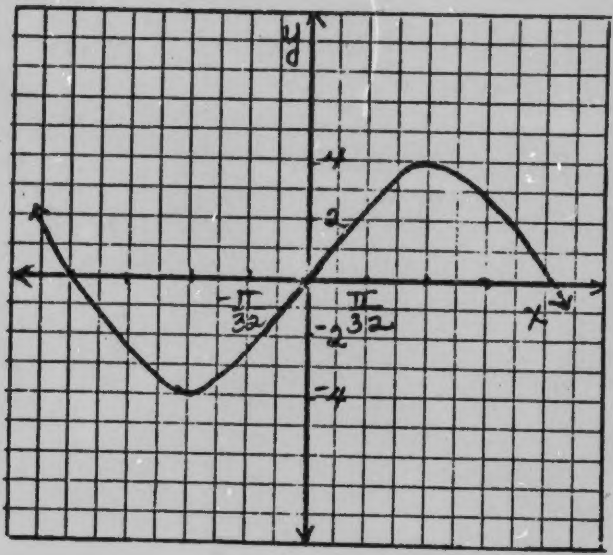
   c) period

   d) boundedness

   e) asymptotes



3. From the graph shown, indicate the properties of continuity, even or odd functions, boundedness, and periodicity.



4. Given the graph of a secant function as shown, indicate the properties of this function:

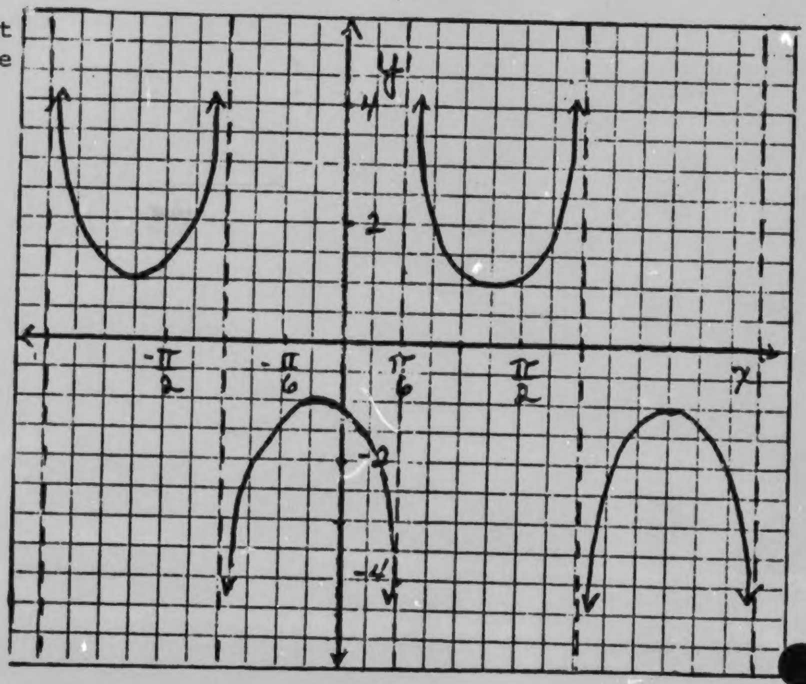
   a) even or odd function

   b) continuous

   c) bounded

   d) period

   e) indicate an interval for which f(x) is a decreasing function over $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$.



2

Pre-Calculus

Unit IV

IV-8 Given an angle, determine its reference angle.

Determine the reference angle for each of the following angles:

| | Angle | Reference Angle |
|---|---|---|
| 1. | | |
| | a) 207° | a) ________ |
| | b) 2.1456 | b) ________ |
| | c) 1813° | c) ________ |
| | d) 5.9143 | d) ________ |
| 2. | | |
| | a) 134° | a) ________ |
| | b) 6.8255 | b) ________ |
| | c) 5.0000 | c) ________ |
| | d) 1400° | d) ________ |
| 3. | | |
| | a) -2.5000 | a) ________ |
| | b) 2893° | b) ________ |
| | c) -253° | c) ________ |
| | d) 12.5000 | d) ________ |

IV-9 Determine the functional values of the special angles (multiples of $\frac{\pi}{6}, \frac{\pi}{4}$).

1. Evaluate each of the following:

   a) $\sin \frac{3}{4}\pi$

   b) $\cos\left(-\frac{7}{6}\pi\right)$

   c) $\cot\left(\frac{3}{2}\pi\right)$

   d) $\sec\left(\frac{17}{3}\pi\right)$

2. Determine each of the following functional values:

   a) $\left(\tan \frac{\pi}{4}\right) \cdot \left(\sec \frac{2}{3}\pi\right)$

   b) $\sin\left(-\frac{7}{3}\pi\right) + \cos \frac{5}{2}\pi$

   c) $\sin\left(-\frac{17}{6}\pi\right) - 2\cos \frac{\pi}{3}$

   d) $\left(\cot \frac{8}{3}\pi + \sin \frac{5}{2}\pi\right)^2$

3. Determine the value of each of the following:

   a) $2\left(\sin \frac{11}{6}\pi\right) - 3\left(\tan \frac{5}{3}\pi\right)$

   b) $\cot\left(-\frac{\pi}{4}\right) + \sin \frac{7}{6}\pi$

   c) $4\left[\sin \frac{\pi}{2} - \cos \frac{11}{3}\pi\right]$

   d) $\tan 0 + \csc \frac{4}{3}\pi$

IV-10 Determine the measure(s) of a special angle, given the functional value of the angle.

1. Determine all angles whose functional values are given:

   a) $\sin\theta = 1$

   b) $\sec\theta = {}^-\sqrt{2}$

   c) $\tan\theta$ is undefined

   d) $\cos\theta = \frac{\sqrt{3}}{2}$

2. Given the functional value of an angle, determine all the angles:

   a) $\tan\theta = -\frac{\sqrt{3}}{3}$

   b) $\cos\theta = 0$

   c) $\csc\theta = 2$

   d) $\sin\theta$ is undefined

3. Evaluate all the angles whose functional values are given:

   a) $\sin\theta = \frac{{}^-\sqrt{3}}{2}$

   b) $\tan\theta = 1$

   c) $\cos\theta = -1$

   d) $\csc\theta = -2$

IV-11 Use a table to determine the functional value of a given angle, interpolating as necessary.

1. Use the tables to determine the functional value of the given angle (interpolate where necessary):

   a) sin 154° 20' = __________

   b) cos (2.5802) = __________

   c) tan (-100° 15') = __________

   d) sin (-1.1170) = __________

2. Using the tables, determine the functional value of the given angle (interpolate where necessary):

   a) cos (-213° 10') = __________

   b) tan (5.2738) = __________

   c) sin 36° 13' = __________

   d) cos 4.3255 = __________

3. Determine the functional value of the given angle by using the tables (interpolate where necessary):

   a) tan 97° 40' = __________

   b) sin 197° 40' = __________

   c) cos (11.2516) = __________

   d) sin (.4310) = __________

**IV-12** Use a table to determine the measure(s) of an angle, given the functional value of an angle, interpolating as necessary.

1. Determine all angles $\theta$, $0 \leq \theta \leq 2\pi$ by use of the tables. Express all angle measurements in degrees (interpolate as necessary).

   a) $\sin \theta = .2700$

   b) $\cos \theta = -.3773$

   c) $\tan \theta = .8693$

   d) $\sin \theta = -.7786$

2. Use tables to determine all angles $\theta$, $0 \leq \theta \leq 2\pi$ whose functional value is given. Express all angle measurements in radians (interpolate as necessary):

   a) $\cos \theta = -.7234$

   b) $\tan \theta = 1.8040$

   c) $\sin \theta = -.9417$

   d) $\cos \theta = .0410$

3. Determine all angles satisfying the given functional values. Express all angle measurements in degrees (interpolate as necessary).

   a) $\sin 0 = .3746$

   b) $\tan \theta = -1.6643$

   c) $\cos \theta = .6450$

   d) $\sin \theta = -.5500$

IV-12 (continued)

4. This type of problem must be done if Objective 22 is covered.

   Determine all values for x satisfying the given equations:

   a) $\cos\left[\frac{\pi}{3}(t - 2)\right] = .2560$

   b) $\sin\left[\frac{\pi}{4}(t + 1)\right] = -\ .4384$

   c) $\cos\left[\frac{\pi}{6}(t - 1)\right] = -\ .3907$

   d) $\sin\left[\frac{\pi}{12}(t - 5)\right] = .8571$

**IV-13** Given the equation or graph of a circular function, determine the period, amplitude, phase shift, and vertical shift.

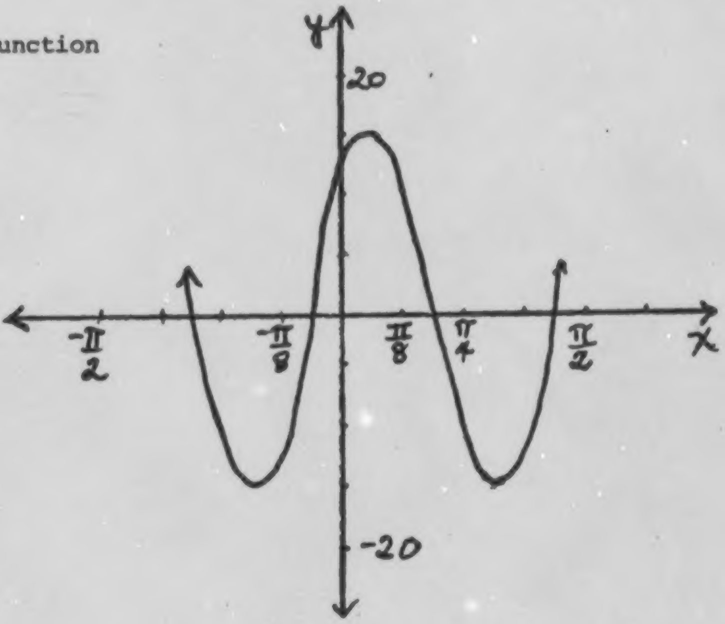
1. From the graph of a sine function shown, determine:

   a) period ____________

   b) amplitude __________

   c) phase shift _______

   d) vertical shift _____

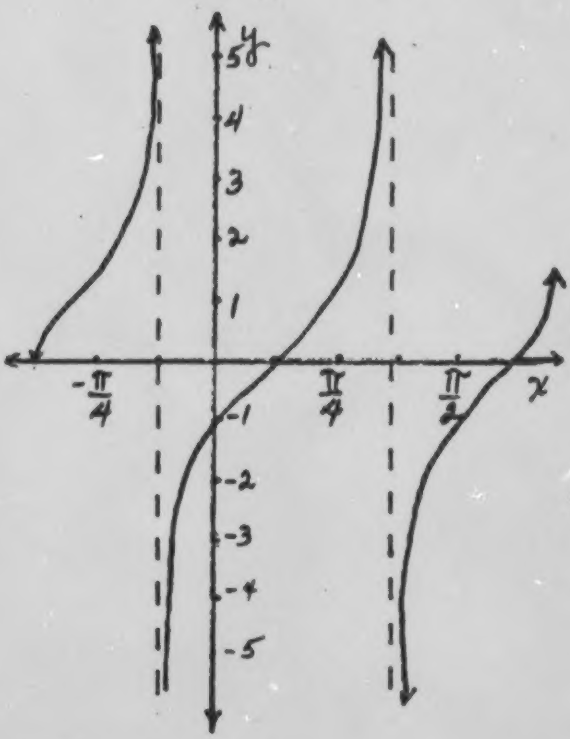


2. From the graph of the tan function as shown, determine

   a) period _____________

   b) phase shift _________

   c) vertical shift _____



IV-13 (continued)

3. Given the equation $y = 4 \cos (2x - 1) + 7$, determine:

   a) period ______________

   b) amplitude ___________

   c) phase shift _________

   d) vertical shift _____

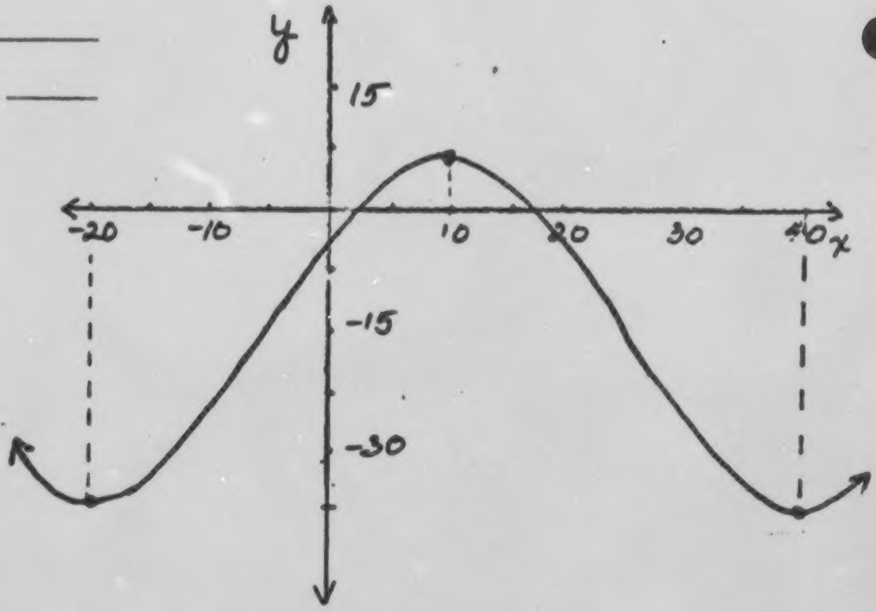
4. Determine each of the following from the cosine function shown:

   a) period ______________

   b) amplitude ___________

   c) phase shift _________

   d) vertical shift _____



IV-14 Sketch the graph of a circular function that has a phase shift and/or vertical shift.

1. Sketch the graph of the function, $f(x) = 4 \cos (2x - \frac{\pi}{2}) - 3$ for $\frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi$.

2. Given $y = 3 \sin (\frac{1}{2} \theta - \pi) + 6$ for $-2\pi \leq \theta \leq 6\pi$, sketch its graph.

3. Sketch the graph of $G(x) = \sec (3x - \frac{\pi}{4}) - 3$ for $-\frac{\pi}{12} \leq \theta \leq \frac{11}{12}\pi$.

4. Given the function $y = .8 \cos [\frac{\pi}{6} (\theta + 2)] + 1.2$, sketch its graph for $-5 \leq \theta \leq 13$.

IV-15 Given the amplitude, period, phase shift, and vertical shift, write the equation of a sine or cosine function.

1. Write an equation of a sine curve with the given characteristics:

   Amplitude: 2; Period: $\pi$; Phase Shift: $\frac{3\pi}{4}$; Vertical Shift: -1

2. Write an equation of a sine curve with the given characteristics:

   Amplitude: $\frac{3}{2}$; Period: $\frac{\pi}{2}$; Phase Shift: $-\frac{\pi}{4}$; Vertical Shift: $\frac{1}{2}$

3. Write an equation of a cosine curve with the given characteristics:

   Amplitude: $\frac{1}{2}$; Period: $\frac{3\pi}{4}$; Phase Shift: $\frac{\pi}{2}$; Vertical Shift: +2

4. Write an equation of a cosine curve with the given characteristics:

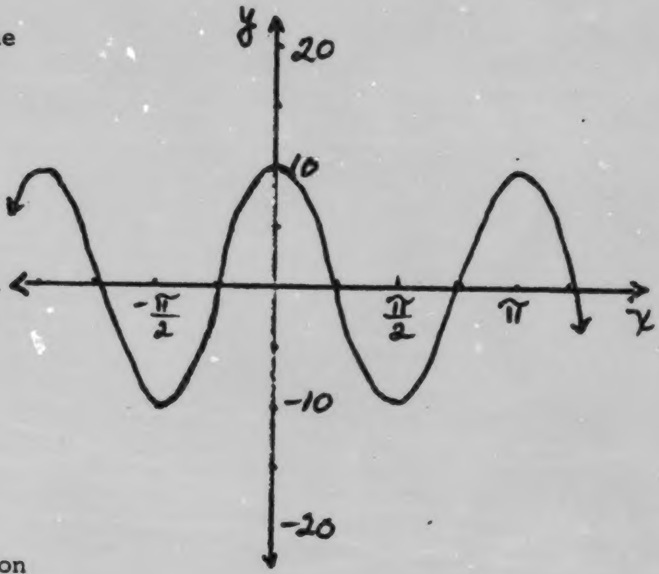
   Amplitude: 3; Period: $3\pi$; Phase Shift: $\frac{3\pi}{2}$; Vertical Shift: $-\frac{5}{2}$

Pre-Calculus

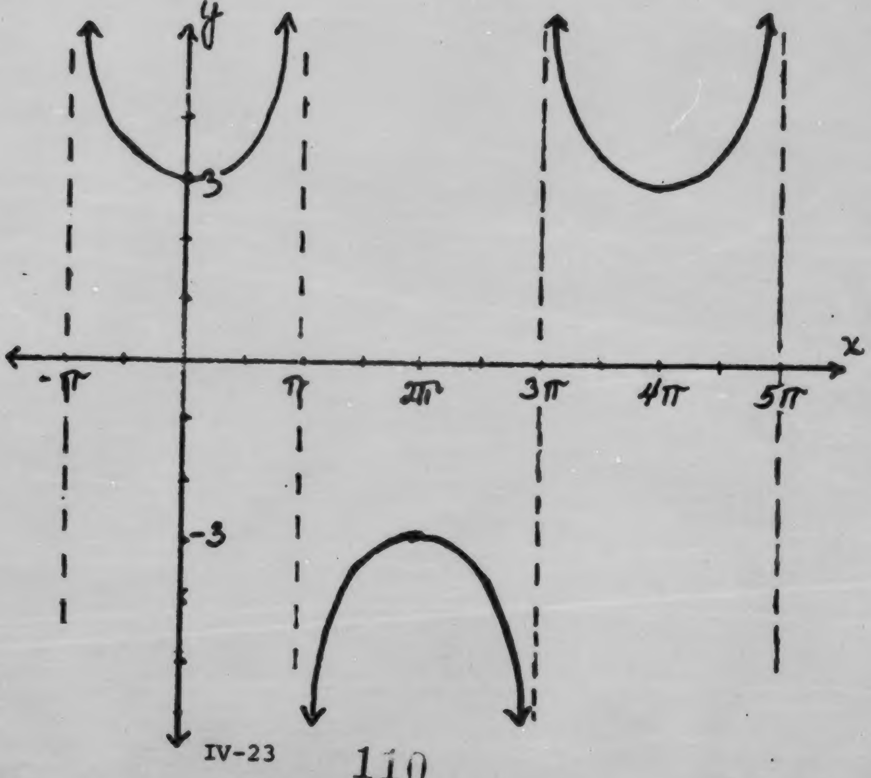
Unit IV

IV-16 Given the graph of a circular function, determine its equation.

1. Determine the equation of the sine function shown at the right.



2. The graph of a secant function is shown. What is its equation?

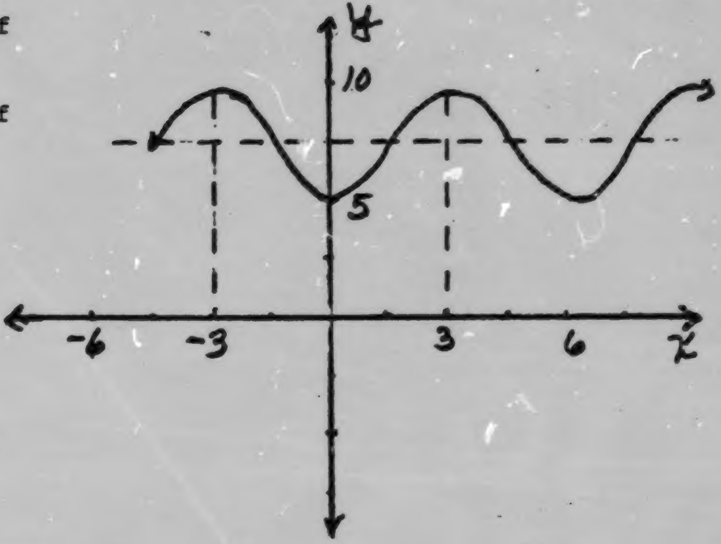


IV-16 (continued)

3.

a. What is the equation of this cosine function?

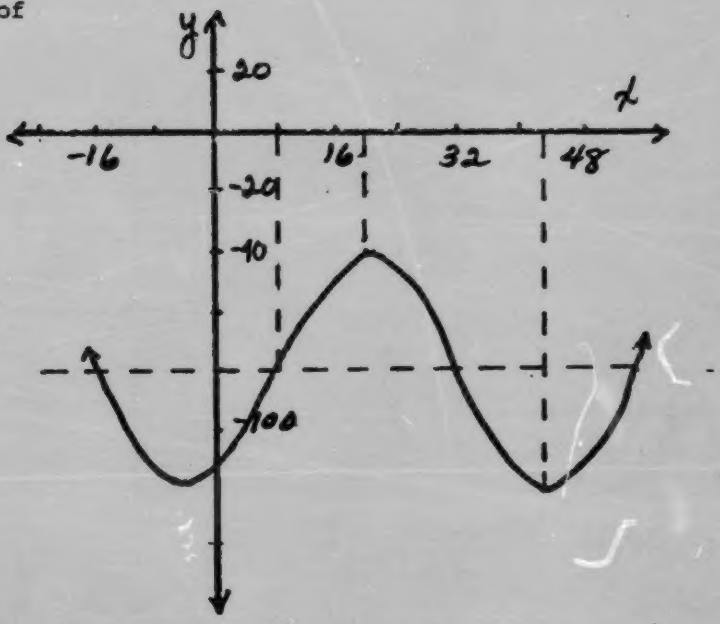
b. What is the equation of this sine function?



4.

a. Determine the equation of the sine function.

b. Determine the equation of the cosine function.



**IV-17** Given two circular functions, f and g, sketch the graph of $f + g$ by addition of ordinates (graphical addition).

1. Given $f(x) = \sin x$ and $g(x) = -2 \cos x$, sketch the graph of f and g on the same set of axes and then sketch $f + g$. Label each graph and make the final graph clearly distinguishable from f and g.

2. Sketch the graph of $h(x) = \sin x + \sin 2x$ by the method of addition of ordinates. Let $f(x) = \sin x$ and $g(x) = \sin 2x$. Label f, g, and h.

3. Given $f(x) = 4 \cos \frac{1}{2}x$ and $g(x) = 3 \sin 2x$, sketch the graph of $f + g$ by addition of ordinates (graphical addition). Label f, g, and $f + g$. Darken the graph of $f + g$.

**IV-18** Solve verbal problems involving uniform circular motion.

1. A point is moving along a wheel of radius 3 units at a constant velocity of $\frac{\pi}{6}$ revolutions per second. Find the length of the arc traveled in 2 seconds.

2. The earth makes one revolution every 24 hours. Determine the angular velocity of the earth in radians per hour.

3. A phonograph record of radius 3 inches revolves on a turntable at a rate of 45 revolutions per minute. What speed are points on the record passing beneath the needle when the needle is one inch from the center?

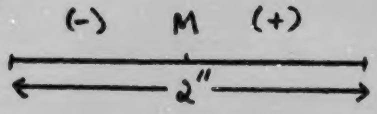
4. A point $P_1$ is located at $(\frac{7}{2}, \frac{7\sqrt{3}}{2})$ when $t = 0$. If $P_1$ is moving with a constant rotational velocity w around a circle with center at the origin, write an expression to represent the coordinates of P at any time t.

IV-19 Solve verbal problems involving simple harmonic motion.

1. A simple harmonic motion is given by the equation $d = \sin \frac{5}{6} \pi t$. Determine the period and frequency of this harmonic motion, where t is in seconds.

2. A wheel revolves at a constant speed of 40 revolutions per second. What is the period of this motion?

3. Point M is the midpoint of a segment 2 inches long. A point moves along this segment with a simple harmonic motion that has a 12 second period. When t = 0, the displacement from M is zero, and when t = 1, the displacement from M is positive. Determine the displacement (negative to left of M and positive to the right) from M at the end of:



a) $\frac{1}{2}$ period

b) $\frac{5}{12}$ period

c) $1 \frac{7}{12}$ periods

d) 21 seconds

IV-20 Solve verbal problems involving other applications of the circular functions (alternating current, radio, Fourier coefficients).

1. The electricity in your home is called "alternating current." The usual house current is "60-cycle" current, which means a frequency of 60 cycles per second. If the quantity of electric current can be represented by

$$I = a \sin wt,$$

how can I be expressed in terms of t if the amplitude is 10 and the current is 60-cycle?

2. Scientists have recently developed a theory that a person's biological functioning is controlled by three factors that vary sinusoidally with time. These biorhythms are physical with a period of 23 days, emotional with a period of 28 days, and intellectual with a period of 33 days. On a particular day, suppose all three of the rhythms are at a high point having an amplitude of 1. Sketch all three cycles for the next 33 days.

IV-21 Construct the equation of a sinusoidal function given data from a set of physical science observations.

1. Your distance d from the ground varies sinuosoidally with time as you ride in a ferris wheel. Suppose the lowest point of the ferris wheel is 5 feet above the ground and the wheel has a diameter of 50 feet. It is also known that the wheel makes one revolution in 12 seconds. At $t = 0$ you are at a height d above the ground. Two seconds later you have reached the top of the wheel. Sketch a graph of this sinusoid and write an equation for this sinusoid.

FOR OTHER SIMILAR PROBLEMS REFER TO: Trigonometry: Functions and Applications, by Foerster. Addison-Wesley

IV-22 Use the equations constructed in Objective 21 to make predictions.

1. Refer to Sample Problem 1 for Objective 21.

   a) From the information given, determine your distance from the ground when t = 0.

   b) When is the second time you are 35 feet above the ground?

Pre-Calculus

Unit IV

## ANSWERS

### IV-1

1.

   a) $\frac{7}{36}\pi$

   b) $-\frac{371}{90}\pi$

   c) 105°

   d) 154° 42' (to nearest degree)

2.

   a) $\frac{41}{36}\pi$

   b) $\frac{49}{720}\pi$

   c) 25° 19' (to nearest degree)

   d) 91° 40' (to nearest degree)

3.

   a) .0548

   b) 180°

   c) 28° 39' (to nearest degree)

   d) .7872

### IV-2

1. 280 inches

2. true

3. $\frac{28}{3}$ radians

4. 3

### IV-3

1. (0, -1)

2. $(\frac{-\sqrt{3}}{2}, \frac{-1}{2})$

3. $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

4. (-1, 0)

### IV-4

Refer to textbook.

### IV-5

1. D: $\{x: x \neq \frac{\pi}{2} + \pi K\}$

   R: R

2. D: R
   R: $-1 \leq f(x) \leq 1$

3. D: R
   R: $-1 \leq f(x) \leq 1$

4. D: $\{x: x \neq \pi K$, K is the integer$\}$

   R: $f(x) \geq 1$ or $f(x) \leq -1$

### IV-6

Refer to textbook.

Pre-Calculus

Unit IV

ANSWERS

## IV-7

1.
   - a) continuous over R
   - b) symmetric with respect to origin
   - c) bounded above and below
   - d) period of $2\pi$
   - e) increasing $-\frac{\pi}{2} \leq x \leq 0$

2.
   - a) odd function
   - b) increasing
   - c) period of $2\pi$
   - d) unbounded
   - e) asymptotes at $x = \pi + 2\pi K$

3.
   - a) continuous
   - b) odd function
   - c) bounded above and below
   - d) period of $\frac{\pi}{4}$

4.
   - a) Function is neither odd nor even.
   - b) Function is discontinuous at $\frac{\pi}{6} + \frac{\pi}{2} K$
   - c) Function is unbounded.
   - d) period of $\pi$
   - e) decreasing over $\frac{-\pi}{12} \leq x \leq \frac{\pi}{6}$

## IV-8

1.
   - a) 27°
   - b) .9956
   - c) 13°
   - d) .3689

2.
   - a) 46°
   - b) .5423
   - c) 1.2876
   - d) 40°

3.
   - a) .6416
   - b) 13°
   - c) 73°
   - d) .0664

## IV-9

1.
   - a) $\frac{\sqrt{2}}{2}$
   - b) $-\frac{\sqrt{3}}{2}$
   - c) 0
   - d) 2

Pre-Calculus

Unit IV

**ANSWERS**

## IV-9

2.

a) $-2$

b) $\frac{-\sqrt{3}}{2}$

c) $-\frac{3}{2}$

d) $\frac{4-2\sqrt{3}}{3}$

3.

a) $-1+3\sqrt{3}$

b) $-\frac{3}{2}$

c) $2$

d) $\frac{-2\sqrt{3}}{3}$

## IV-10

1.

a) $\theta = \frac{\pi}{2} + 2\pi K$

b) $\theta = \frac{3}{4}\pi + 2\pi K$

$\theta = \frac{5}{4} + 2\pi K$

c) $0 = \frac{\pi}{2} + \pi K$

d) $\theta = \frac{\pi}{6} + 2\pi K$

$\theta = \frac{\pi}{6} + 2\pi K$

## IV-10 (continued)

2.

a) $\theta = \frac{5}{6}\pi + 2\pi K$

$\theta = \frac{11}{6}\pi + 2\pi K$

b) $\theta = \frac{\pi}{2} + \pi K$

c) $\theta = \frac{\pi}{6} + 2\pi K$

$\theta = \frac{5}{6}\pi + 2\pi K$

d) $\phi$

3.

a) $\theta = \frac{4}{3}\pi + 2\pi K$

$\theta = \frac{5}{3}\pi + 2\pi K$

b) $\theta = \frac{\pi}{4} + \pi K$

c) $\theta = \pi + 2\pi K$

d) $\theta = \frac{7}{6}\pi + 2\pi K$

$\theta = \frac{11}{6}\pi + 2\pi K$

## IV-11

1.

a) .4331

b) $-.8465$

c) 5.5301

d) $-.8988$

## IV-11

2.

a) - .8371

b) - 1.5901

c) .5908

d) - .3773

3.

a) - 7.4287

b) - .3035

c) .2532

d) .4178

## IV-12

1.

a) $\theta = 15° \ 40'$ or $164° \ 20'$

b) $\theta = 112° \ 10'$ or $247° \ 50'$

c) $\theta = 41°$ or $221°$

d) $\theta = 231° \ 8'$ or $308° \ 52'$

2.

a) $\theta = 2.3795$ or $3.9037$

b) $\theta = 1.0646$ or $4.2062$

c) $\theta = 4.3692$ or $5.0556$

d) $\theta = 1.5298$ or $4.7534$

## IV-12 (continued)

3.

a) $\begin{cases} \theta = 22° + 360 \ K° \\ \theta = 158° + 360 \ K° \end{cases}$

b) $\theta = 121° + 180 \ K°$

c) $\begin{cases} \theta = 49° \ 50' + 360 \ K° \\ \theta = 310! \ 10' + 360 \ K° \end{cases}$

d) $\begin{cases} \theta = 213° \ 22' + 360 \ K° \\ \theta = 326° \ 38' + 360 \ K° \end{cases}$

4.

a) (using radians)

$t = 3.2528 + 6n;$ $n \in$ Integers

$t = .7472 + 6 n$

b) $t = 3.5778 + 8n:$ $n \in$ Integers

$t = -1.5778 + 8n$

c) $t = 4.7666 + 12n:$ $n \in$ Integers

$t = 9.2334 + 12n$

d) $t = 8.9332 + 24n:$ $n \in$ Integers

$t = 13.0669 + 24n$

## IV-13

1.

a) $\frac{\pi}{2}$

b) 15

c) $\frac{-\pi}{16}$

d) 0

2.

a) $\frac{\pi}{2}$

b) $\frac{\pi}{8}$

c) 0

3.

a) $\pi$

b) 4

c) $\frac{1}{2}$

d) moved up 7 units

4.

a) 60

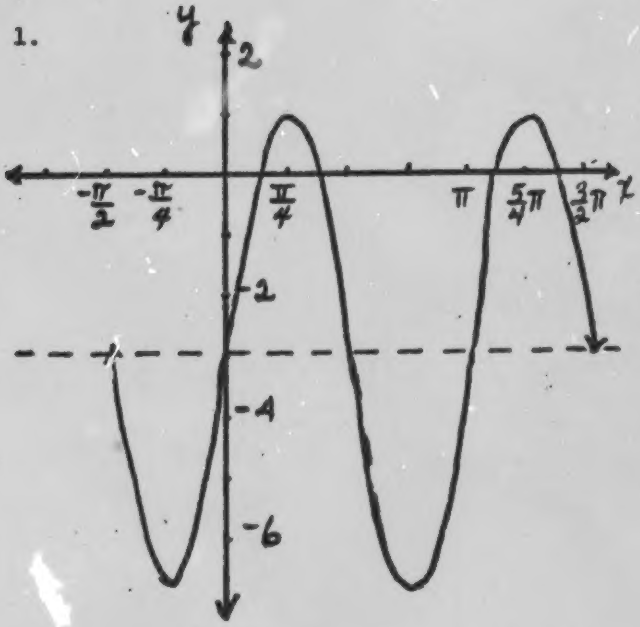
b) $22\frac{1}{2}$

c) 10 units to right

d) moved down 15 units

## IV-14

1.



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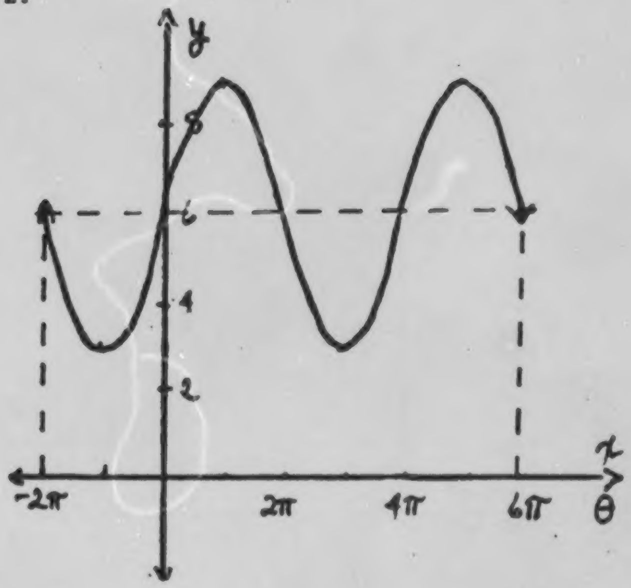


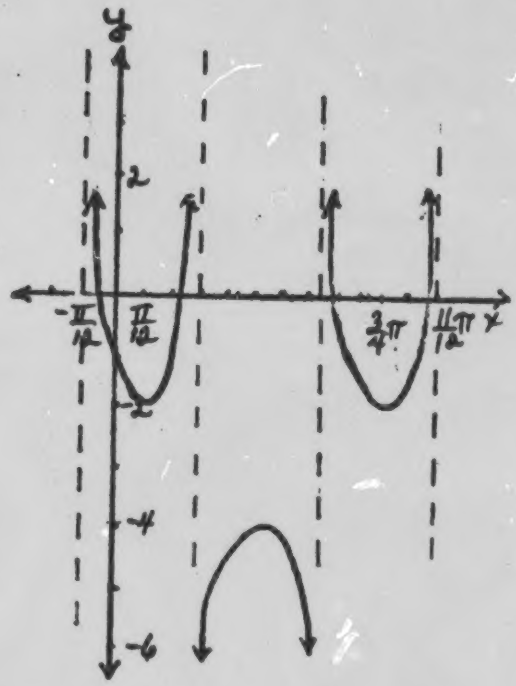
Pre-Calculus

Unit IV Circular Functions

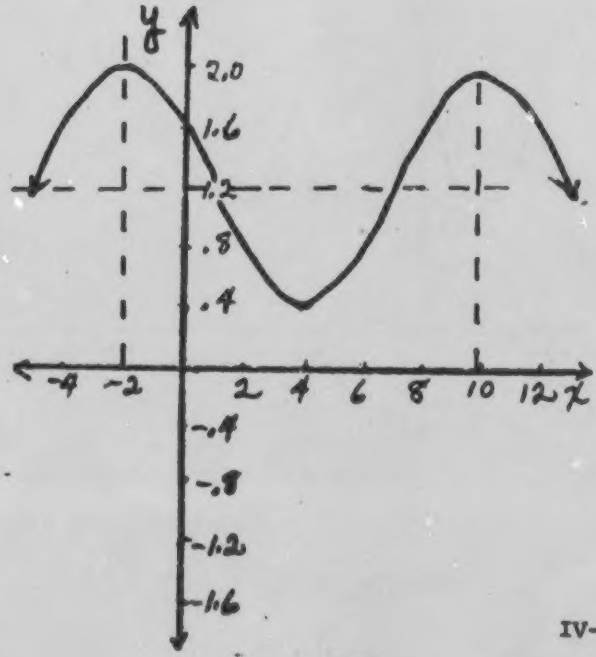
## ANSWERS

### IV-14

3.



4.



### IV-15

1. $y = 2\,(2 \sin x - \frac{3\pi}{2}) - 1$

2. $y = \frac{3}{2}\,(4 \sin x + \frac{\pi}{2}) + \frac{1}{2}$

3. $y = \frac{1}{2}\,(\frac{8}{3} \cos x - \frac{3\pi}{16}) + 2$

4. $y = 3\,(\frac{2}{3} \cos x - \pi) - \frac{5}{2}$

Pre-Calculus

Unit IV

ANSWERS

IV-16

1. $y = 10 \sin(2x + \frac{\pi}{2})$

2. $y = 3 \sec (\frac{1}{2} x)$

3.

a) $y = 2.5 \cos [\frac{\pi}{3} (x + 3)] + 7.5$

b) $y = 2.5 \sin [\frac{\pi}{3} (x + 4.5)] + 7.5$

Answers may vary.

4.

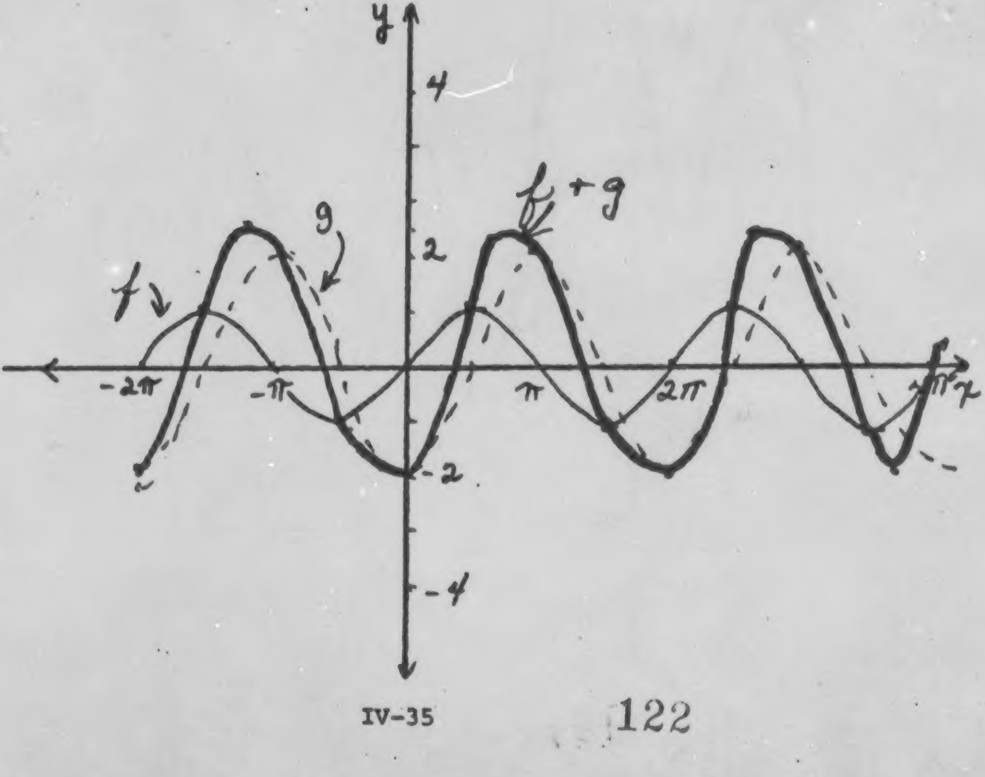
a) $y = 40 \sin [\frac{\pi}{24} (x - 8)] - 80$

b) $y = 40 \cos [\frac{\pi}{24} (x - 20)] - 80$

Answers may vary.

IV-17

1.



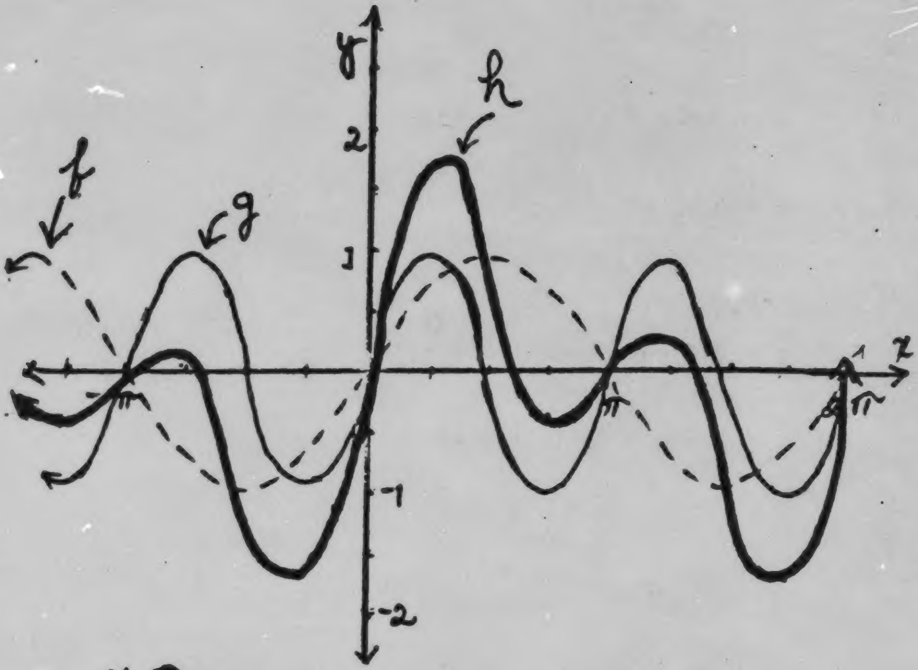
**Pre-Calculus**

**Unit IV**

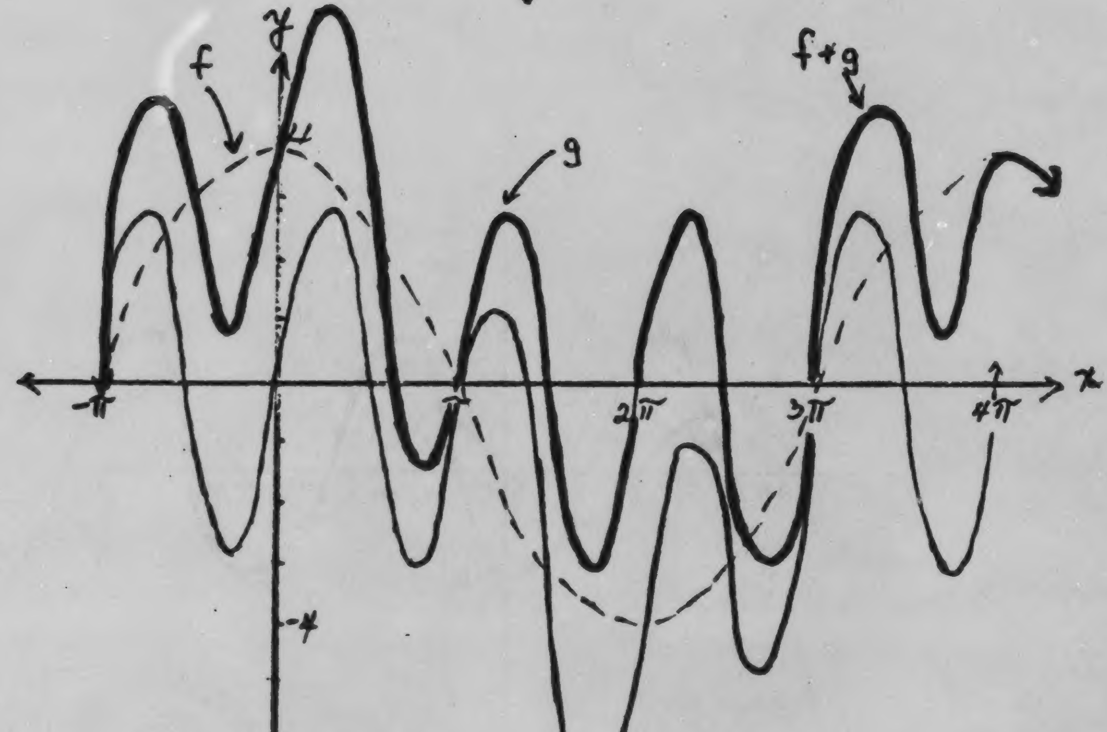
**ANSWERS**

**IV-17**

2.



3.



Pre-Calculus

Unit IV

ANSWERS

**IV-18**

1. $\pi$

2. $\frac{\pi}{12}$ radians/hour

3. $90\pi$ inches/minute

4. $x = 7 \cos (wt + \frac{\pi}{3})$: $y = 7 \sin (wt + \frac{\pi}{3})$

**IV-19**

1. 2.4 seconds

   $\frac{5}{12}$ cycles per second

2. $\frac{1}{40}$ second

3.

   a) 0

   b) $\frac{\sqrt{3}}{2}$

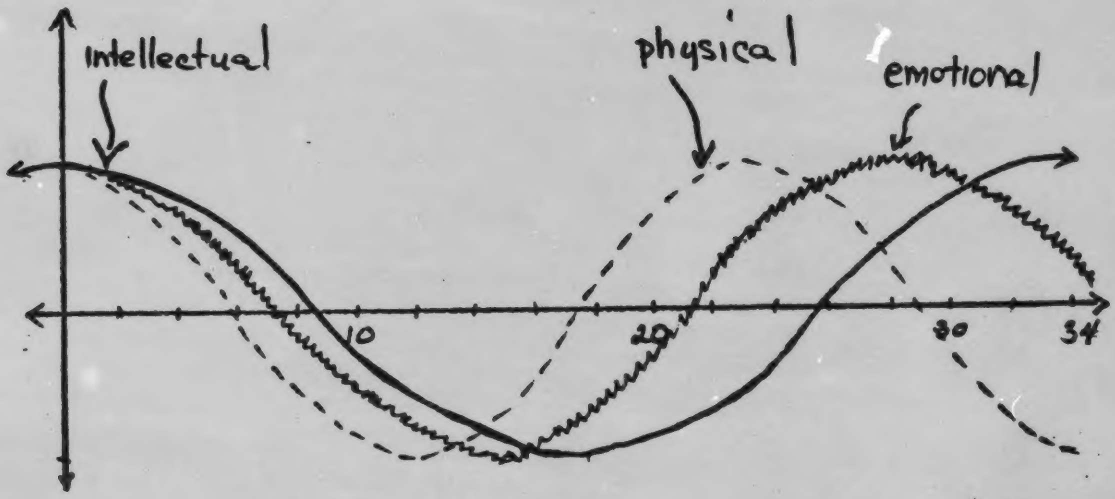
   c) $\frac{-\sqrt{3}}{2}$

   d) 1

**IV-20**

1. $I = 10 \sin 120 \pi t$

2.



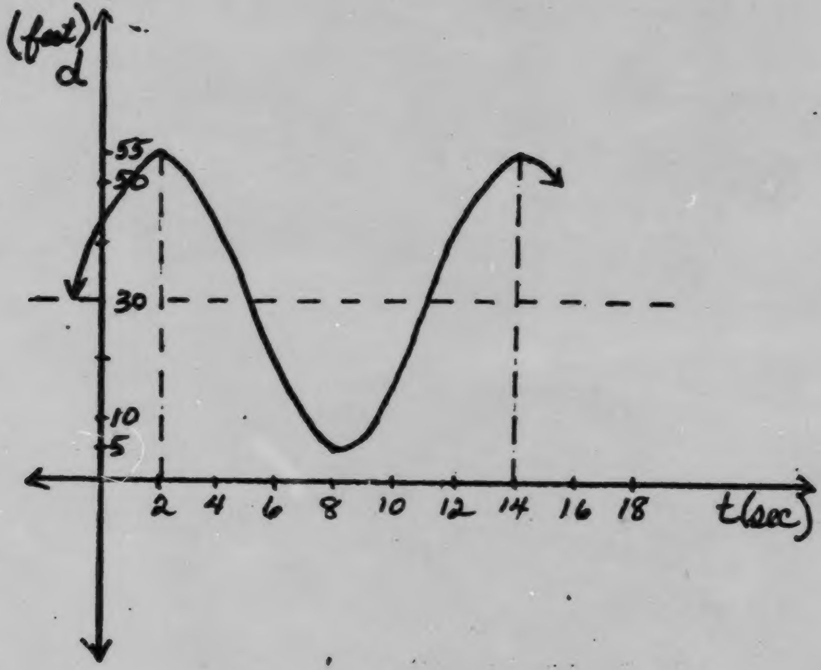
Pre-Calculus

Unit IV

**ANSWERS**

IV-21

1. $d = 25 \cos [\frac{\pi}{6} (t - 2)] + 30$



IV-22

1.

a) d = 42.5 feet.

b) The second time you are 35 feet above the ground is when t = 11.3844 seconds. (This problem may be done in terms of degrees instead of radians.)

## Unit V Trigonometric Identities

### Overview:

This unit will introduce to the student the many trigonometric identities and formulas that are needed to understand the intricate relationships that exist between the six circular functions. The completion of the numerous proofs should greatly enhance the student's mathematical sophistication. The treatment here includes the product-to-sum and the sum-to-product identities because of their use in Calculus.

### Suggestions to the Teacher

For motivational purposes, many of the objectives include numerical problems so that the student can observe how the functional values of certain nonstandard angles can be determined. Because of this, some Pre-Calculus textbooks may need to be supplemented. Excellent reference textbooks would be:

Trigonometry: Functions and Applications, Foerster (1977)

Modern Algebra and Trigonometry - Structure and Method - Book 2, Dolciani, et al: (1982)

Often the question of the amount of memorization of formulas is brought up with regard to the teaching of this unit. It is recommended that all basic identities (Pythagorean, e.g., $\sin^2\theta + \cos^2\theta = 1$, etc.; quotient; reciprocal; negative; and cofunction) be memorized. The sum (and difference) and double angle formulas for sine, cosine, and tangent need to be memorized as well as the half angle formulas for sine and cosine. In addition, it is expected that students should be able to derive these formulas and show the logical sequence of the derivation of the formulas. The half angle formula for tangent and the product-to-sum and sum-to-product formulas need not be memorized but the student should be able to derive such formulas.

### Suggested Time

10 days

## Unit V Trigonometric Identities

### PERFORMANCE OBJECTIVES

1. Given the functional value, determine other functional values by applying the Pythagorean identities.
2. Prove identities using the reciprocal, quotient, and Pythagorean identities.
3. Show the derivation for the formula for cos $(\alpha - \beta)$ or cos $(\alpha + \beta)$. (operational)
4. Apply the sum and difference formulas to prove identities.
5. Apply the sum and difference formulas to determine the functional value of a given angle.
6. Given functional values of two angles, evaluate sum and difference formulas.
7. Prove the double angle formulas for sin 2x, cos 2x, and tan 2x.
8. Prove the half angle formulas for sin $\frac{1}{2}x$, cos $\frac{1}{2}x$, and tan $\frac{1}{2}x$.
9. Prove identities using double and/or half angle formulas.
10. Apply the double and half angle formulas to determine the functional value of a given angle.
11. Given the functional value of an angle, evaluate double and half angle formulas.
12. Verify identities using the sum-to-product and/or product-to-sum formulas.
13. Apply the sum-to-product and product-to-sum formulas to convert from one form to another, evaluating where applicable.

Unit V - Trigonometric Identities

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 109 | | | | 272 | |
| 2 | 105-109 | 142-144 | | 157-162 | | |
| 3 | 135-138 | 145 | | 163-164 | 288-289 | |
| 4 | 138-139<br>143-144 | 145-150 | | 163-167 | 289, 291 | |
| 5 | 139, 143 | 149 | | 168-170 | 292 | |
| 6 | 139 | 149 | | 168-170 | | |
| 7 | 144-145 | 150 | | 171-173 | 290 | |
| 8 | 146 | 151 | | 171-173 | 290-291 | |
| 9 | 148 | 152-153 | | 173-174 | | |
| 10 | 147 | 152 | | 173 | 292 | |
| 11 | 147-148 | 152 | | 173 | 292 | |
| 12 | 150 | 155 | | 167 | | |
| 13 | 149-150 | 154-155 | | | | |

V-1 Given the functional value, determine other functional values by applying the Pythagorean identities.

1. Given $\cos x = \frac{-\sqrt{3}}{2}$ in the second quadrant, find $\sin x$ by using $\sin^2 x + \cos^2 x = 1$.

2. Given $\tan x = \frac{\sqrt{3}}{3}$ in the third quadrant, find $\sec x$ by using $\tan^2 x + 1 = \sec^2 x$.

3. Given $\csc x = -\sqrt{2}$ in the fourth quadrant, find $\cot x$ by using $1 + \cot^2 x = \csc^2 x$.

4. Given $\sec x = \frac{41}{9}$ in the first quadrant, find $\tan x$ by using $\tan^2 x + 1 = \sec^2 x$.

V-2 Prove identities using the reciprocal, quotient, and Pythagorean identities.

Prove each of the following identities:

1. $(1 + \sec x)(\sec x - 1) = \frac{\sin x \cdot \sec x}{\cos x \cdot \csc x}$

2. $\frac{\tan x + \sin x}{1 + \cos x} = \tan x$

3. $\csc x + \cot x = \frac{\sin x}{1 - \cos x}$

4. $\frac{\tan x - \sin x}{\tan x \cdot \sin x} = \frac{\tan x \cdot \sin x}{\tan x + \sin x}$

Pre-Calculus

Unit V

V-3 Show the derivation for the formula cos $(\alpha - \beta)$ or cos $(\alpha + \beta)$.

Show the derivation for:

1. $\cos(\alpha - \beta) = \cos\alpha \; \cos\beta + \sin\alpha \; \sin\beta$

2. $\cos(\alpha + \beta) = \cos\alpha \; \cos\beta - \sin\alpha \; \sin\beta$

V-4 Apply the sum and difference formulas to prove identities.

Prove each of the following identities:

1. $\cot(\alpha + \beta) = \dfrac{\cot\alpha \cdot \cot\beta - 1}{\cot\alpha + \cot\beta}$

2. $\sec(\alpha + \beta) = \dfrac{\sec\alpha \cdot \sec\beta}{1 - \tan\alpha \; \tan\beta}$

3. $\tan(\frac{3\pi}{2} - x) = \cot x$

4. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan\alpha \; \tan\beta}{1 + \tan\alpha \; \tan\beta}$

V-5 Apply the sum and difference formulas to determine the functional value of a given angle.

1. Evaluate cos 285° using the cos $(\alpha+\beta)$ formula. Leave your result in simplest radical form.

2. Evaluate sin 255° using the sin $(\alpha-\beta)$ formula. Leave your result in simplest radical form.

3. Evaluate tan (-195°) using the tan $(\alpha+\beta)$ formula. Leave your result in simplest radical form.

4. Evaluate tan (15°) using the tan $(\alpha-\beta)$ formula. Leave your result in simplest radical form.

V-6 Given functional values of two angles, evaluate sum and difference formulas.

1. $\angle A$ and $\angle B$ are in the first quadrant, and $\cos A = \frac{3}{5}$, $\cos B = \frac{5}{13}$. Find $\cos (A + B)$.

2. $\cos A = \frac{-4}{5}$ and $\angle A$ is in second quadrant; $\sin B = \frac{8}{17}$ and $\angle B$ is in first quadrant, Find $\sin (A - B)$.

3. $\angle A$ and $\angle B$ are in the first quadrant, and $\cos A = \frac{7}{25}$, $\sin B = \frac{3}{5}$. Find $\tan (A + B)$.

4. $\angle A$ and $\angle B$ are in the third quadrant, and $\cos A = -\frac{5}{13}$ and $\cos B = \frac{-4}{5}$. Find $\tan (A - B)$.

V-7 Prove the double angle formulas for sin 2x, cos 2x, and tan 2x.

Prove each of the following identities:

1. $\sin 2x = 2 \sin x \cos x$

2. $\cos 2x = \cos^2 x - \sin^2 x$

3. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

V-8 Prove the half angle formulas for $\sin \frac{1}{2} x$, $\cos \frac{1}{2} x$, and $\tan \frac{1}{2} x$.

Prove the following:

1. $\sin \frac{1}{2} x = \pm \sqrt{\frac{1}{2} (1 - \cos x)}$

2. $\cos \frac{1}{2} x = \pm \sqrt{\frac{1}{2} (1 + \cos x)}$

3. $\tan \frac{1}{2} x = \frac{\sin x}{1 + \cos x}$

Pre-Calculus

Unit V

**V-9** Prove identities using double and/or half angle formulas.

Prove the following:

1. $\cos 3x = 4\cos^3 x - 3\cos x$
2. $\dfrac{1 + \cos 2x}{\sin 2x} = \cot x$
3. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$
4. $\csc \frac{1}{2}x = \pm \dfrac{\sqrt{2 + 2\cos x}}{\sin x}$

**V-10** Apply the double and half angle formulas to determine the functional value of a given angle.

1. Use the half angle formula to evaluate sin 75°. Leave result in simplest radical form.
2. Use the half angle formula to evaluate $\tan 112\frac{1}{2}°$. Leave result in simplest radical form.
3. Use the double angle formula to evaluate $\tan \frac{2\pi}{3}$.
4. Use the double angle formula to evaluate $\cos \frac{5\pi}{3}$.

## Unit V Trigonometric Identities

**V-11** Given the functional value of an angle, evaluate double and half angle formulas.

1. If $\cos x = -\frac{3}{5}$, find $\sin 2x$ if $\pi < x < \frac{3\pi}{2}$.

2. If $\sin x = \frac{4}{5}$, find $\tan \frac{x}{2}$ if $\frac{\pi}{2} < x < \pi$.

3. If $\sin x = -\frac{7}{25}$, find $\sin \frac{x}{2}$ if $\frac{3\pi}{2} < x < 2\pi$.

4. If $\tan x = \frac{3}{4}$, find $\cos \frac{x}{2}$ if $\pi < x < \frac{3\pi}{2}$.

**V-12** Verify identities using the sum-to-product and/or product-to-sum formulas.

1. Prove $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan\left(\frac{x-y}{2}\right)$

2. Prove $\frac{\cos 6x + \cos 4y}{\sin 6x - \sin 4y} = \cot x$.

3. Prove $\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2y + \cos 3x} = \tan 2$.

4. Prove: $\frac{\sin(2x-y) + \sin y}{\cos(2x-y) + \cos y} = \tan x$.

V-13 Apply the sum-to-product and product-to-sum formulas to convert from one form to another, evaluating where applicable.

1. Find the exact value of $\sin 75° + \sin 15°$ by using the sum-to-product formula.

2. Find the exact value of $\cos 165° - \cos 75°$ by using the sum-to-product formula.

3. Find the exact value of $\cos 45° \cdot \sin 15°$ by using the product-to-sum formula.

4. Find the exact value of $\sin 225° \cdot \sin 15°$ by using the product-to-sum formula.

**ANSWERS**

V-1

1. $\frac{1}{2}$

2. $-\frac{2\sqrt{3}}{3}$

3. $-1$

4. $\frac{40}{9}$

V-2

1. $(1 + \sec x)(\sec x - 1) = \sec x - 1 + \sec^2 x - \sec x$

$$= \sec^2 x - 1$$

$$= \tan x^2$$

$$= \frac{\sin x \bullet \sin x}{\cos x \bullet \cos x}$$

$$= \frac{\sin x \bullet \sec x}{\cos x \bullet \csc x}$$

2. $\frac{\tan x + \sin x}{1 + \cos x}$

$$= \frac{\frac{\sin x}{\cos x} + \sin x}{1 + \cos x}$$

$$= \frac{\frac{\sin x + \sin x \cos x}{\cos x}}{1 + \cos x}$$

$$= \frac{\sin x\,(1 + \cos x)}{\text{cox } x\,(1 + \cos x)}$$

$$= \tan x$$

Pre-Calculus

Unit V

ANSWERS

V-2

3. $\csc x + \cot x$

$$= \frac{1}{\sin x} + \frac{\cos x}{\sin x}$$

$$= \frac{1 + \cos x}{\sin x} \cdot \left[\frac{1 - \cos x}{1 - \cos x}\right]$$

$$= \frac{\sin^2 x}{\sin x\,(1 - \cos x)}$$

$$= \frac{\sin x}{1 - \cos x}$$

4. $\frac{\tan x - \sin x}{\tan x \sin x}$

$$= \frac{\tan x - \sin x}{\tan x \sin x} \cdot \left[\frac{\tan x + \sin x}{\tan x + \sin x}\right]$$

$$= \frac{\tan^2 x - \sin^2 x}{\tan x \sin x\,(\tan x + \sin x)}$$

$$= \frac{\frac{\sin^2 x}{\cos^2 x} - \sin^2 x}{\frac{\sin x}{\cos x} \circ \sin x\,(\tan x + \sin x)}$$

$$= \frac{\frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x} \cdot \frac{\cos x}{\sin^2 x}}{\tan x + \sin x}$$

$$= \frac{\frac{1 - \cos^2 x}{\cos x}}{\tan x + \sin x}$$

$$= \frac{\frac{\sin^2 x}{\cos x}}{\tan x \sin x}$$

$$= \frac{\tan x \sin x}{\tan x + \sin x}$$

V-3 Refer to approved texts.

Pre-Calculus

Unit V

ANSWERS

V-4

1. $\cot(\alpha + \beta) = \dfrac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)}$

$$= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} + \dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$$

$$= \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

2. $\sec(\alpha + \beta) = \dfrac{1}{\cos(\alpha + \beta)}$

$$= \frac{1}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{1}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\sec\alpha\sec\beta}{1 - \tan\alpha\tan\beta}$$

Pre-Calculus

Unit V

ANSWERS

V-4

3. $$\tan\left(\frac{3\pi}{2} - x\right) = \frac{\sin\left(\frac{3\pi}{2} - x\right)}{\cos\left(\frac{3\pi}{2} \quad x\right)}$$

$$= \frac{\sin\frac{3\pi}{2}\cos x - \cos\frac{3\pi}{2}\sin x}{\cos\frac{3\pi}{2}\cos x + \sin\frac{3\pi}{2}\sin x}$$

$$= \frac{-\cos x}{-\sin x} = \cot x$$

4. $$\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{1 - \tan\alpha\tan\beta}{1 + \tan\alpha\tan\beta}$$

V-5

1. $\frac{\sqrt{6} - \sqrt{2}}{4}$

2. $\frac{-\sqrt{2} - \sqrt{6}}{4}$

3. $\sqrt{3} - 2$

4. $2 - \sqrt{3}$

Pre-Calculus

Unit V

ANSWERS

V-6

1. $-\frac{33}{65}$

2. $\frac{77}{85}$

3. $-\frac{117}{44}$

4. $\frac{33}{56}$

V-7 Refer to approved texts.

V-8 Refer to approved texts.

V-9

1. $\cos 3x = \cos(x + 2x)$

$$= \cos x \cos 2x - \sin x \sin 2x$$

$$= \cos x (\cos^2 x - \sin^2 y) - \sin x (2 \sin x \cos x)$$

$$= \cos^3 x - \cos x \sin^2 x - 2 \cos x \sin^2 x$$

$$= \cos^3 x - 3 \cos x \sin^2 x$$

$$= \cos^3 x - 3 \cos x (1 - \cos^2 x)$$

$$= \cos^3 x - 3 \cos x + 3 \cos^3 x$$

$$= 4 \cos^3 x - 3 \cos x$$

Pre-Calculus

Unit V

ANSWERS

V-9:

2. $$\frac{1 + \cos 2x}{\sin 2x} = \frac{1 + \cos^2 x - \sin^2 x}{2 \sin x \ \cos x}$$

$$= \frac{1 + \cos^2 x - (1 - \cos^2 x)}{2 \sin x \ \cos x}$$

$$= \frac{1 + \cos^2 x - (1 - \cos^2 x)}{2 \sin x \ \cos x}$$

$$= \frac{2 \cos^2 x}{2 \sin x \cos x}$$

$$= \frac{\cos x \ \cos x}{\sin x \ \cos x}$$

$$= \cot x$$

3. $$\sec 2x = \frac{1}{\cos 2x}$$

$$= \frac{1}{\cos^2 x - \sin^2 x}$$

$$= \frac{1}{\cos^2 x - (1 - \cos^2 x)}$$

$$= \frac{1}{2 \cos^2 x - 1}$$

$$= \frac{\frac{1}{\cos^2 x}}{\frac{2 \cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}}$$

$$= \frac{\sec^2 x}{2 - \sec^2 x}$$

**ANSWERS**

V-9

4. $$\csc \frac{1}{2}x = \frac{1}{\sin \frac{1}{2}x}$$

$$= \frac{1}{\pm\sqrt{\frac{1}{2}(1 - \cos x)}}$$

$$= \pm \sqrt{\frac{2}{1 - \cos x}} = \pm\sqrt{\frac{2\ (1 + \cos x)}{(1 - \cos x)\ (1 + \cos x)}}$$

$$= \pm\sqrt{\frac{2\ (1 + \cos x)}{1 - \cos^2 x}}$$

$$= \pm \frac{\sqrt{2 + 2 \cos x}}{\sin x}$$

V-10

1. $\frac{\sqrt{2 + \sqrt{3}}}{2}$

2. $-\sqrt{2} - 1$

3. $-\sqrt{3}$

4. $\frac{1}{2}$

ANSWERS

V-11

1. $-\frac{24}{25}$

2. $2$

3. $\frac{\sqrt{2}}{10}$

4. $\frac{\sqrt{10}}{10}$

V-12

1. $$\frac{\sin x - \sin y}{\cos x + \cos y} = \frac{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

$$= \tan\left(\frac{x-y}{2}\right)$$

2. $$\frac{\cos 6x + \cos 4x}{\sin 6x - \sin 4x} = \frac{2\cos 5x \cos x}{2\cos 5x \sin x}$$

$$= \cot x$$

3. $$\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \frac{\sin 3x + \sin x + \sin 2x}{\cos 3x + \cos x + \cos 2x}$$

$$= \frac{2\sin 2x \cos x + \sin 2x}{2\cos 2x \cos x + \cos 2x}$$

$$= \frac{\sin 2x\,(2\cos x + 1)}{\cos 2x\,(2\cos x + 1)}$$

$$= \tan 2x$$

4. $$\frac{\sin(2x - y) + \sin y}{\cos(2x - y) + \cos y} = \frac{2\sin x \cos(x - y)}{2\cos x \cos(x - y)}$$

$$= \tan x$$

Pre-Calculus

Unit V

**ANSWERS**

## V-13

1. $\sin 75^\circ + \sin 15^\circ = 2 \sin \frac{90^\circ}{2} \cos \frac{60^\circ}{2}$

$$= 2 \sin 45^\circ \cos 30^\circ$$

$$= 2 \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{6}}{2}$$

2. $\cos 165^\circ - \cos 75^\circ = -2 \sin \frac{240^\circ}{2} \sin \frac{90^\circ}{2}$

$$= -2 \sin 120^\circ \sin 45^\circ$$

$$= -2 \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{6}}{2}$$

3. $\cos 45^\circ \sin 15^\circ = \frac{1}{2} \sin 60^\circ - \frac{1}{2} \sin 30^\circ$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{4}$$

4. $\sin 225^\circ \sin 15^\circ = -\frac{1}{2} \cos 240^\circ + \frac{1}{2} \cos 210^\circ = -\frac{1}{2} \cdot -\frac{1}{2} + \frac{1}{2} \cdot -\frac{\sqrt{3}}{2}$

$$= \frac{1 - \sqrt{3}}{4}$$

## Unit VI Inverse Circular Functions and Trigonometric Equations

### Overview

The calculus demands as much understanding of the inverse circular functions as of circular functions themselves. The ability to manipulate, evaluate, show, and prove identities involving inverse circular functions is thus deemed as important as any of the other course objectives. Of course, solving trigonometric equations using inverse concepts and notation continues to be a fundamental objective.

### Suggestions to the Teacher

In this unit, in particular, the various approved texts show marked differences in approach and emphasis. For this reason it is strongly recommended that the cross-reference key be used in advance to preview and select those text materials considered best for one's class.

The following texts are suggested for use with this unit:

Advanced Mathematics. Coxford (provides a good coverage of all objectives including trigonometric inequalities)

As a reference these textbooks are suggested:

Algebra, Trigonometry, and Analytic Geometry. Rees (strong in Objectives 9 and 10)

Trigonometry: Functions and Applications. Foerster (strong in Objectives 1-4, 10, and practical problems)

### Time

8-10 days (2 weeks)

## Unit VI Inverse Circular Functions and Trigonometric Equations

### PERFORMANCE OBJECTIVES

1. Determine the domain and range for each of the inverses of the circular functions.

2. Graph the inverses of the circular functions.

3. State the domain and the range for each of the inverse circular functions.

4. Sketch the graphs of the inverse circular functions.

5. Determine the set of angles that satisfies a given trigonometric expression containing inverse relation notation.

6. Determine the angle (principal value) that satisfies a given trigonometric expression containing inverse function notation.

7. Evaluate trigonometric expressions containing inverse function notation.

8. Solve equations involving inverse circular functions.

9. Verify identities or statements containing inverse circular functions.

10. Solve trigonometric equations containing one or more circular function.

### Optional

11. Solve trigonometric inequalities.

Unit VI - Inverse Circular Functions and Trigonometric Equations

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 174-175 | 156 | | 186-189 | | |
| 2 | 175 | 156 | | 186-189 | | |
| 3 | 174-176 | 157 | | 186-189 | 320-324 | |
| 4 | 175-177 | 156 | | 186-189 | 321-324 | |
| 5 | 172-174 | 157-158 | | 191-192 | | |
| 6 | 179 | 157-158 | | 191-192 | 321-322, 326 | |
| 7 | 173-174<br>178-180 | 157-158 | | 192 | 321-324, 326 | |
| 8 | | 161 | | 192 | | |
| 9 | 178, 180 | 159 | | 192 | 327 | |
| 10 | 181<br>183-184 | 159-160 | | 193-195 | 276 | |
| 11 | 185-187 | 160 | | | | |

VI-1 Determine the domain and range for each of the inverses of the circular functions.

1. Determine the domain and range of the inverse relation x = sin y.

2. Determine the domain and range of the inverse relation x = cot y.

3. Given the function f(x) = cos x, determine:

   a) the domain of the inverse of f(x)

   b) the range of the inverse of f(x)

4. Given the function f(x) = csc x, determine:

   a) the domain of the inverse of f(x)

   b) the range of the inverse of f(x)

VI-2 Graph the inverses of the circular functions.

1. Sketch the graph of the inverse relation of x = cos y.

2. Sketch the graph of the inverse relation x = tan y.

3. Given f(x) = sin x, sketch the graph of the inverse of f(x).

4. Given f(x) = sec x, sketch the graph of the inverse of f(x).

VI-3 State the domain and the range for each of the inverse circular functions.

1. State the domain and range of $y = \text{Sin}^{-1}x$.

2. State the domain and range of $y = \text{Tan}^{-1}x$.

3. State the domain and range for the function $f(x) = \text{Arccos } x$.

4. State the domain and range for the function $f(x) = \text{Arccot } x$.

VI-4 Sketch the graphs of the inverse circular functions.

1. Sketch the graph of the function $y = \text{Sin}^{-1}x; \ -1 \le x \le 1$.

2. Sketch the graph of the function $y = \text{Cos}^{-1}x = -1 \le x \le 1$.

3. Graph the function $f(x) = \text{Arctan } x$.

4. Graph the function $f(x) = \text{Arcsec } x$.

VI-5 Determine the set of angles that satisfies a given trigonometric expression containing inverse relation notation.

1. Determine the set of angles given by: $\{y: y = \arcsin(-\frac{1}{2})\}$.

2. Determine the set of angles given by: $\{y: y = \arctan\sqrt{3}\}$.

3. Determine, in degrees, the angle(s) represented by: arccos (-1).

4. Determine, in degrees, the angle(s) represented by: arcsec 1.743.

VI-6 Determine the angle (principal value) that satisfies a given trigonometric expression containing inverse function notation.

1. State the angle represented by: Arccos $\frac{\sqrt{3}}{2}$.

2. State the angle represented by: $\mathrm{Tan}^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

3. Determine the principal values of the inverse circular functions given below:

   a) $y = \mathrm{Arccos}\ 0$

   b) $y = \mathrm{Arctan}\ (-1)$

   c) $y = \mathrm{Arcsin}\ (-1)$

   d) $y = \mathrm{Arcsec}\ 0$

4. Determine, to the nearest 10 minutes, the principal values of the inverse circular functions given below:

   a) $y = \mathrm{Cos}^{-1}\ (.1076)$

   b) $y = \mathrm{Tan}^{-1}\ (-.1405)$

   c) $y = \mathrm{Sec}^{-1}\ (-4.945)$

VI-7 Evaluate trigonometric expressions containing inverse function notation.

1. Evaluate: $\cos\left(\text{Arcsin } \frac{\sqrt{3}}{2}\right)$.

2. Evaluate: $\sin\ (\text{Arctan } \frac{5}{12})$.

3. Evaluate: $\cot\ (\text{Arcsin } u)$.

4. Evaluate: $\sec\ (\text{Sin}^{-1}\ .4384)$.

5. Evaluate: $\tan\left(\text{Arccos } \frac{1}{2} - \text{Arcsec } \frac{2}{\sqrt{3}}\right)$.

VI-8 Solve equations involving inverse circular functions.

1. Solve for x in terms of y: $2y = 3\ \text{Sin}^{-1} x - 5$.

2. Solve for x in terms of y: $y = 6 + 2 \tan \frac{\pi}{5} (x - 3)$.

3. Solve for x: $\text{Arccos } (2x^2 - 2x) = \frac{2\pi}{3}$.

4. Solve for x: $\frac{3\pi}{4} = \tan^{-1} (3x^2 - 4x)$.

5. Solve for x: $\text{Sin}^{-1} x - \text{Cos}^{-1} x = \frac{\pi}{2}$.

VI-9 Verify identities or statements containing inverse circular functions.

1. Show that $\sin(\text{Arcsin } x + \text{Arcsin } y) = x\sqrt{1 - x^2} + y\sqrt{1 - x^2}$.

2. Show that $\cos(\sin^{-1} x + \cos^{-1} x) = 0$.

3. Verify the following: $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \frac{\pi}{4}$.

4. Verify the following: $2 \text{ Arctan } \frac{2}{3} = \text{Arctan } \frac{12}{5}$.

VI-10 Solve trigonometric equations containing one or more circular functions.

1. Determine all values that satisfy the equation: $3 \sec x - 11 = -5$.

2. Solve for x: $\cos 2x = 1 - \sin x;\ 0 \leq x < 2\pi$.

3. Solve for x over the interval $0 \leq x < 2\pi$ : $2 \cos^2 x - \cos x = 1$.

4. Determine the value(s) of x to the nearest 10' over the interval $0° \leq x < 360°$: $2 \tan x + \sec x = 1$.

**VI-11 Solve the trigonometric inequalities.**

1. Determine the solution set of the following: $\{x : 2 \sin x - 1 \geq 0; 0 \leq x < 2\pi\}$.

2. Determine the solution set of the following: $\{x : 2 \sin^2 x - \cos x - 1 \geq 0; 0 \leq x < 2\pi\}$.

Pre-Calculus

Unit VI

## ANSWERS

### VI-1

1. Domain: $\{x : -1 \leq x \leq 1\}$

   Range: $\{y : y \in \text{reals}\}$

2. Domain: $\{x : x \in \text{reals}\}$

   Range: $\{y : y \neq \kappa \bullet \pi\}$ ($\kappa \in$ integers)

3. Domain: $\{x : -1 \leq x \leq 1\}$

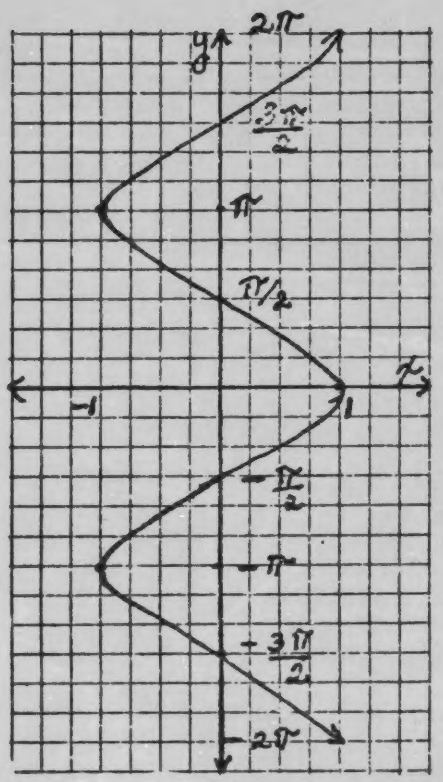
   Range: $\{y : y \in \text{reals}\}$

4. Range: $\{y : y \neq \kappa \bullet \pi\}$ ($\kappa \in$ integers)

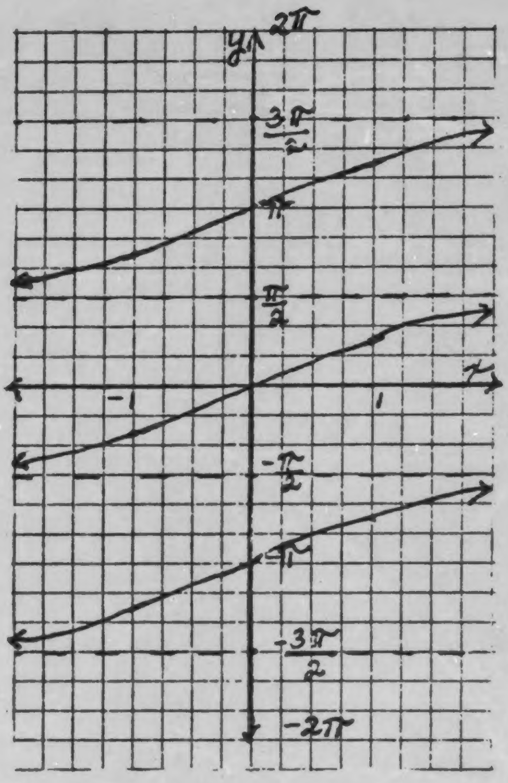
   Domain: $\{x : x \geq 1\}$ $\{x : x \leq -1\}$

### VI-2

1.



2.

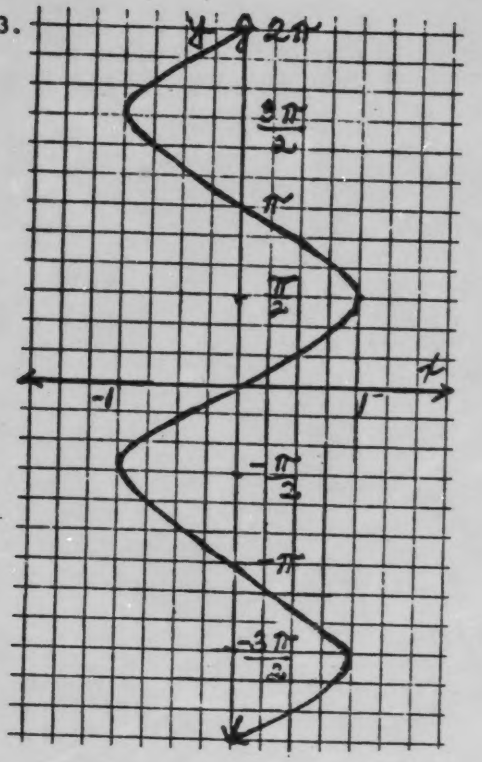


Pre-Calculus

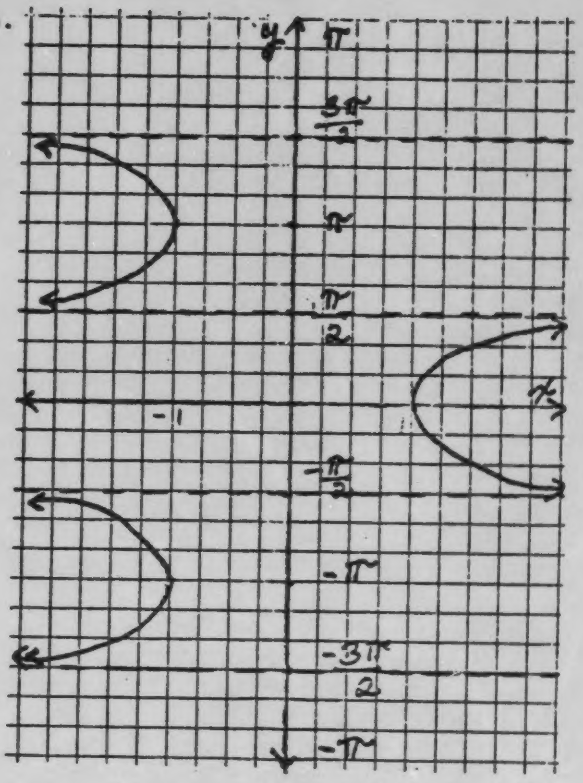
Unit VI

ANSWERS

3.



4.



VI-3

1. **Domain:** $\{x : -1 < x \le 1\}$

   **Range:** $\{y : \frac{-\pi}{2} \le y - \frac{\pi}{2}\}$

2. **Domain:** $\{x : x \in \text{reals}\}$

   **Range:** $\{y : -\frac{\pi}{2} < y < \frac{\pi}{2}\}$

3. **Domain:** $\{x : -1 \le x \le 1\}$

   **Range:** $\{y : 0 \le y \le \pi\}$

4. **Domain:** $\{x : x \in \text{reals}\}$

   **Range:** $\{y : 0 < y < \pi\}$

Pre-Calculus

Unit VI

ANSWERS

VI-4

1. Refer to any approved text
2. Refer to any approved text
3. Refer to any approved text
4. Refer to any approved text

VI-5

1. $\{\frac{7\pi}{6} + 2\kappa\pi; \frac{11\pi}{6} + 2\kappa\pi\}$
2. $\{\frac{\pi}{3} + \kappa\pi\}$
3. $\{(2\kappa + 1)\,180^\circ\}$
4. $\{55^\circ + 2\kappa\pi; 305^\circ + 2\kappa\pi\}$

VI-6

1. $\frac{\pi}{6}$
2. $\frac{-\pi}{6}$
3. a $\frac{\pi}{2}$
   b $\frac{-\pi}{4}$
   c $\frac{-\pi}{2}$
   d undefined
4. a $83^\circ\ 50'$
   b $-8^\circ\ 00'$
   c $101^\circ\ 40'$

Pre-Calculus

Unit VI

**ANSWERS**

VI-7

1. $\frac{1}{2}$

2. $\frac{5}{13}$

3. $\frac{\sqrt{1 - u^2}}{u}$

4. 1.113

5. $\frac{\sqrt{3}}{3}$

VI-8

1. $x = \sin\left(\frac{2y + 5}{3}\right)$

2. $x = 3 + \frac{5}{\pi} \tan^{-1}\left(\frac{y - 6}{2}\right)$

3. $x = \frac{1}{2}$

4. $x = \frac{1}{3}, 1$

5. $x = 1$

VI-9

1. $\sin(\text{Arcsin } x + \text{Arcsin } y)$

$= \sin(\text{Arcsin } x)\cos(\text{Arcsin } y) + \cos(\text{Arcsin } x)\sin(\text{Arcsin } y)$

$= x \cdot \sqrt{1 - y^2} + \sqrt{1 - x^2} \cdot y$

$= x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$

## ANSWERS

### VI-9

2. $\cos(\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x)$

$= \cos(\mathrm{Sin}^{-1}x)\cos(\mathrm{Cos}^{-1}x) - \sin(\mathrm{Sin}\text{-}1x)\sin(\mathrm{Cos}^{-1}x)$

$= \sqrt{1-x^2} \bullet x - x \bullet \sqrt{1-x^2}$

$= 0$

3. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \frac{\pi}{4}$

$\tan(\mathrm{Tan}^{-1}\frac{1}{3} + \mathrm{Tan}^{-1}\frac{1}{2}) = \tan(\frac{\pi}{4})$

$$\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\bullet\frac{1}{2}} = 1$$

$$\frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$$1 = 1$$

4. $2 \text{ Arctan } \frac{2}{3} = \text{Arctan } \frac{12}{5}$

$\tan(2 \text{ Arctan } \frac{2}{3}) = \tan(\text{Arctan } \frac{12}{5})$

$$\frac{2\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)^2} = \frac{12}{5}$$

$$\frac{\frac{4}{3}}{\frac{5}{9}} = \frac{12}{5}$$

$$\frac{12}{5} = \frac{12}{5}$$

**Pre-Calculus**

**Unit VI**

## ANSWERS

### VI-10

1. $\{\frac{\pi}{3} + 2k\pi;\ \frac{5\pi}{3} + 2k\pi\}$

2. $\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}\}$

3. $\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$

4. $\{0°, 233°\ 10'\}$

### V-11

1. $\{x : \frac{\pi}{6} \le x \le \frac{5\pi}{6}\}$

2. $\{x : \frac{\pi}{3} \le x \le \frac{5\pi}{3}\}$

## Unit VII Applications of Trigonometric Functions

### Overview

Determining the solutions to triangles is the emphasis of this unit. Trigonometric functions are first applied to solving right triangles and are then employed in the development of the Laws of Sines/Cosines to include oblique triangles. Formulas for the areas of triangles, sectors, and segments are also presented.

### Suggestions to the Teacher

A great deal of computational time may be saved if the use of calculators is permitted; therefore, Objectives 2 and 3 are optional, depending on the availability of calculators.

The students should be required to derive the Law of Sines and the Law of Cosines. When solving oblique triangles, it is advisable for the students to draw a sketch with the given conditions. Even though the Law of Tangents is only briefly mentioned in some of the textbooks, it does provide an additional method for solving triangles, with or without logarithms. Later the laws may be incorporated in formulating the equations for determining the area of a triangle. (Only one version of the area formulas is given in Objective 13). The presentation of Hero's formula at this time may be the student's first exposure but the concept is not difficult to understand.

A review from Geometry on determining areas of segments and sectors may precede the discussion of these areas in this unit.

Suggested time: 10 days

Pre-Calculus

## Unit VII Applications of Trigonometric Functions

### PERFORMANCE OBJECTIVES

1. Apply a trigonometric function to determine a side and/or an angle of a right triangle.

2. Use a log-trig table to determine the logarithm of the functional value of a given angle. (optional)

3. Use a log-trig table to determine the measure of an angle, given the logarithm of the functional value of an angle. (optional)

4. Solve verbal problems involving right triangles. (optional: computation with logarithms)

5. Show a derivation of the Law of Sines.

6. Given two angles and a side of an oblique triangle, (A.A.S. or A.S.A), apply the Law of Sines to determine the missing side opposite one of the given angles.

7. Determine the number of solutions in an oblique triangle, given two sides and an angle opposite one of them (the ambiguous case).

8. Given two sides and an angle opposite one of them (S.S.A.) in an oblique triangle, apply the Law of Sines to determine the acute (or obtuse) angle opposite the other side (the ambiguous case).

9. Show a derivation of the Law of Cosines.

10. Given two sides and the included angle (S.A.S.) of an oblique triangle, apply the Law of Cosines to determine the third side.

11. Given three sides of an oblique triangle (S.S.S.), apply the Law of Cosines to determine a specified angle.

12. Given two sides and the included angle (S.A.S.) of an oblique triangle, apply the Law of Tangents to determine the other two angles. (optional)

13. Apply the formula $K = \frac{1}{2}ab \text{ Sin}C$ to determine the area of an oblique triangle.

14. Apply the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ to determine the area of an oblique triangle. (s = semi-perimeter)

15. Apply the formula $A = \frac{1}{2}r^2\theta$ ($\theta$ expressed in radians) to determine the area of a sector.

16. Apply the formula $A = \frac{1}{2}r^2(\theta - \sin\theta)$ ($\theta$ expressed in radians) to determine the area of a segment. (optional)

Unit VII - Applications of Trigonometric Functions

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | | 172-174 | | 136-138 | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | 173-174 | | 137-138 | | |
| 5 | 156-157 | 176 | | 139 | | |
| 6 | 158-159<br>162-166 | 176-180 | | 140, 142 | | |
| 7 | 162-164 | 177-180 | | 140, 142 | | |
| 8 | 162-165 | 177-180 | | 142 | | |
| 9 | 160-161 | | | 140-141 | | |
| 10 | 161<br>164-166 | 178-180 | | 141-142 | | |
| 11 | 164-166 | 178-180 | | 141-142 | | |
| 12 | | 180 | | | | |
| 13 | 167-170 | 180<br>183-184 | | 142 | | |
| 14 | 168-170 | 183 | | | | |
| 15 | | 182-184 | | | | |
| 16 | | 182-184 | | | | |

VII-1 Apply a trigonometric function to determine a side and/or an angle of a right triangle.

1. Given right triangle ABC with right angle at C, if m ∠ A = 20° and b = 5.0, then a =

   a) 13.8 b) 1.8 c) 1.7 d) 4.7 e) none of these

2. Given right triangle ABC with right angle at C, if c = 15.6 and b = 6.8, determine m ∠ B.

   a) 66° 24' b) 64° 5' c) 25° 51' d) 23° 36' e) none of these

3. The angle of elevation of the top of a building when viewed from the street is 65°. The distance of the observer to the building is 80 m. What is the height of the building?

   a) 171 m b) 72.5 m c) 37.3 m d) 33.8 m e) none of these

VII-2 Use a log-trig table to determine the logarithm of the functional value of a given angle.

1. If m ∠ A = $18^\circ$ 20', determine log sin A.

2. Determine log cos $37^\circ$ 13'.

3. Determine log tan $64^\circ$ 46'.

VII-3 Use a log-trig table to determine the measure of an angle, given the logarithm of the functional value of an angle.

1. If log cos A = 9.9260-10, determine m ∠ A.

2. If log sec A = 0.5259, determine m ∠ A.

3. If log cot A = 9.4200-10, determine m ∠ A to the nearest minute.

VII-4 Solve verbal problems involving right triangles. (optional; computation with logarithms)

1. A 25-foot ladder leans against a house with the foot of the ladder 6.4 feet from the side of the house. What angle does the ladder make with the ground? Determine your answer to the nearest degree.

2. An airplane flies on a compass heading of 140° 32' at 625 mph. How far south and how far east of the starting point is the plane after 2 hours? Determine your answers to the nearest tenth of a mile.

3. An observer in a lighthouse 40 m above the surface of the ocean measures an angle of depression at 0° 54' to a distant ship. How many kilometers is the ship from the base of the lighthouse? Determine your answer to the nearest hundredth of a kilometer.

VII-5 Show a derivation of the Law of Sines.

1. Derive the Law of Sines.

2. Show a derivation of the Law of Sines.

VII-6 Given two angles and a side of an oblique triangle (A.A.S. or A.S.A.), apply the Law of Sines to determine the missing side opposite one of the given angles.

1. In $\Delta ABC$, $m\angle B = 62° 0'$, $m\angle C = 42° 0'$, and $b = 16.0$. Determine c and a to the nearest tenth of a unit.

2. In $\Delta ABC$, $a = 12.0$, $m\angle B = 58° 0'$, and $m\angle C = 26° 0'$. Determine c to the nearest tenth of a unit.

3. Suppose that you are a pilot of a commercial airliner. You find it necessary to detour around a group of thunderstorms. You turn at an angle of 18° to your original path, fly for a while, turn, and intercept your original path at an angle of 32°, 75 kilometers from where you left it. How much further did you have to go because of the detour? Give the answer correct to two significant digits.

VII-7 Determine the number of solutions in an oblique triangle, given two sides and an angle opposite one of them (the ambiguous case).

1. In $\Delta ABC$, $m \angle B = 68^\circ$, $b = 13.1$, and $a = 6.6$.

2. In $\Delta ABC$, $m \angle B = 26^\circ\ 40'$, $b = 60.42$, and $a = 82.44$.

3. In $\Delta ABC$, $m \angle B = 32^\circ\ 10'$, $b = 8.64$, and $a = 17.4$.

VII-8 Given two sides and an angle opposite one of them (S.S.A.) in an oblique triangle, apply the Law of Sines to determine the acute (or obtuse) angle opposite the other side (the ambiguous case).

1. In $\Delta ABC$ $m \angle B = 52^\circ\ 40'$, $b = 1.42$ and $a = 0.554$. Determine $m \angle A$ to the nearest 10 minutes.

2. In $\Delta ABC$ $m \angle B = 37^\circ\ 50'$, $b = 13.8$, and $a = 22.3$. Determine $m \angle A$ to the nearest 10 minutes.

3. In $\Delta ABC$ $m \angle B = 28^\circ\ 34'$, $b = 1464$, and $a = 3142$. Determine $m \angle A$ to the nearest minute.

VII-9 Show a derivation of the Law of Cosines.

1. Derive the Law of Cosines.

2. Show a derivation of the Law of Cosines.

VII-10 Given two sides and the included angle (S.A.S.) of an oblique triangle, apply the Law of Cosines to determine the third side.

1. In $\Delta ABC$, if $b = 5.6$, $c = 7.4$, and $m \angle A = 48^\circ$, determine the third side of the triangle to the nearest tenth.

2. In $\Delta SAT$, if $s = 12.5$, $t = 24.3$, and $m \angle A = 72^\circ\ 40'$, determine the third side of the triangle to the newest tenth.

3. In $\Delta KOA$, $m \angle O = 132^\circ\ 24'$, $k = 18.74$, and $a = 4.213$. Determine the third side of the triangle to 4 significant digits.

**VII-11** Given three sides of an oblique triangle (S.S.S.), apply the Law of Cosines to determine a specified angle.

1. In $\Delta ZAP$, $z = 5.2$, $a = 7.6$, and $p = 8.4$. Determine $m \angle P$.

2. In $\Delta PET$, $p = 34.7$, $e = 25.4$, and $t = 21.3$. Determine $m \angle P$ to the nearest 10 minutes.

3. In $\Delta MAG$, $m = 81.23$, $a = 61.23$, and $g = 46.64$. Determine $m \angle A$ to the nearest minute.

**VII-12** Given two sides and the included angle (S.A.S.) of an oblique triangle, apply the Law of Tangents to determine the other two angles.

1. In $\Delta GMC$, $g = 12$, $m \angle M = 42°$, and $c = 22$. Determine the other two angles to the nearest half degree using the Law of Tangents.

2. In $\Delta MAC$ $m = 52.8$, $m \angle A = 120° 40'$, and $c = 15.2$. Determine the other two angles to the nearest 10 minutes using the Law of Tangents.

VII-13 Apply the formula $K = \frac{1}{2}ab \sin C$ to determine the area of an oblique triangle.

1. Determine the area $\Delta ABC$ if $b = 38$, $c = 46$, and $m\angle A = 68°$ to 2 significant digits.

2. Determine the area of $\Delta GAF$ if $a = 46.8$, $f = 30.4$, and $m\angle G = 100° 10'$ to 3 significant digits.

3. Determine the area of an equilateral triangle whose side is 31.7 to 3 significant digits.

VII-14 Apply the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ to determine the area of an oblique triangle. ($s$ = semi-perimeter)

1. Determine the area of $\Delta ABC$ given $a = 11$, $b = 14$, and $c = 17$.

2. Determine the area of $\Delta TRW$ given $t = 15.3$, $r = 22.5$, and $w = 26.4$ to 3 significant digits.

3. A surveyor measures the three sides of a triangular field and gets 134, 168, and 242 meters. What is the area of the field to 3 significant digits?

**VII-15** Apply the formula $A = \frac{1}{2} r^2\theta$ ($\theta$ expressed in radians) to determine the area of a sector.

1. Determine the area of a sector of a circle with radius $12\frac{1}{2}$ and central angle of $\frac{\pi}{8}$. Leave answer in terms of $\pi$.

2. Determine the area of a sector of a circle with radius 7.20 and central angle of 75°. Use $\pi \doteq 3.14$.

3. A radar antenna turns through a horizontal angle of 80°. If its range is 36.0 km, what area can it sweep? Use $\pi \doteq 3.14$.

**VII-16** Apply the formula $A = \frac{1}{2} r^2 (\theta - \sin\theta)$ ($\theta$ expressed in radians) to determine the area of a segment.

1. Determine area of a segment of a circle with radius $7\frac{1}{2}$ and central angle of $\frac{\pi}{6}$. Leave the answer in terms of $\pi$.

2. Determine the area enclosed by a circle of radius 6 and a regular inscribed hexagon. Leave the answer in terms of $\pi$ and radicals.

Pre-Calculus

Unit VII

ANSWERS

VII-1

1. B

2. C

3. A

VII-2

1. 9.4977-10

2. 9.9011-10

3. .3267

VII-3

1. 32° 30'

2. 72° 40'

3. 75° 16'

VII-4

1. 75° 10'

2. 794.5 mi. east

   965 mi. south

3. 2.55 km

VII-5

See an approved text for an appropriate answer.

VII-6

1. $c = 12.1$, $a = 17.6$

2. $c = 5.3$

3. 7.1 km

Pre-Calculus

Unit VII

ANSWERS

## VII-7

1. 1 solution
2. 2 solutions
3. No solution

## VII-8

1. $m\angle A = 18^\circ\ 00'$
2. $m\angle A = 82^\circ\ 20'$ or $m\angle A = 97^\circ\ 40'$
3. No solution: $\sin\angle A = 1.02626$

## VII-9

See an approved text for an appropriate answer.

## VII-10

1. $a = 5.5$
2. $a = 23.8$
3. $o = 21.80$

## VII-11

1. $m\angle P = 80^\circ$
2. $m\angle P = 95^\circ\ 30'$
3. $m\angle A = 48^\circ\ 28'$

## VII-12

1. $m\angle G = 31.5^\circ$, $m\angle C = 106.5^\circ$
2. $m\angle M = 47^\circ\ 10'$, $m\angle C = 12^\circ\ 10'$

## VII-13

1. 810
2. 700
3. 435

Pre-Calculus

Unit VII

ANSWERS

VII-14

1. 76.7 sq. units
2. 171.8 sq. units
3. 10800

VII-15

1. $\frac{625}{64}\pi$
2. 33.9 sq. units
3. 904 $km^2$

VII-16

1. $\frac{75}{16}(\pi-3)$
2. $6\pi - 9\sqrt{3}$

**Pre-Calculus** - Elementary Functions

## Unit VIII Complex Numbers

### Overview

This unit is a continuation of the student's knowledge of complex numbers, as well as an introduction to the trigonometric (or polar) form of a complex number. Material covered in this unit will also reinforce skills previously learned, such as using tables to determine the trigonometric functional value of an angle and the relationship between the trigonometric functional value of an angle ($\theta$) and its opposite angle ($-\theta$).

### Suggestions to the Teacher

It is expected that a student is able to add, subtract, multiply, and divide complex numbers in rectangular form. However, it may be necessary to review these skills. The objectives used the term "trigonometric" form of a complex number. The word "polar" form can be used interchangeably with the word "trigonometric." The $r \text{ cis } \theta$ notation may be used for $r(\cos \theta + \sin \theta)$.

After the completion of this unit, a student should realize that the trigonometric form of a complex number offers not only another approach to computing with complex numbers but perhaps a simpler approach to the solution of a problem. This can be illustrated by relating skills and concepts previously taught such as the solution of a polynomial equation of the form $x^n = k$ (Unit II) and the development of identities for $\cos 3\theta$ or $\sin 3\theta$ (Unit V).

### Suggested Time

3 days

Pre-Calculus

## Unit VIII Complex Numbers

### PERFORMANCE OBJECTIVES

1. Explain how the complex number $x + yi$ can be written in trigonometric form.
2. Plot complex numbers on the complex plane, given the numbers in rectangular or trigonometric form.
3. Convert complex numbers from the rectangular form to the trigonometric form and vice versa.
4. Use the trigonometric form of complex numbers to determine a product.
5. Use the trigonometric form of complex numbers to determine a quotient.
6. Apply DeMoivre's Theorem to determine a power of a complex number.
7. Apply DeMoivre's Theorem to determine the roots of a complex number.
8. Apply DeMoivre's Theorem to determine all the roots of an equation in the form $x^n = k$, $k$ is a constant.

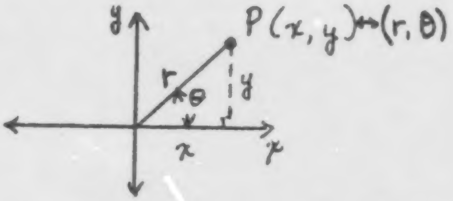
Unit VIII - Complex Numbers

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 187-188 | 186 | | 175-176 | | |
| 2 | 52-54 | 58-60 | | | | |
| 3 | 189-191 | 186 | | 176-177 | | |
| 4 | 189-191 | 187-189 | | 177-178, 180 | | |
| 5 | 190-191 | 187-189 | | 177 | | |
| 6 | 192-196 | 189-193 | | 178-180 | | |
| 7 | 192-196 | 189-193 | | 178-180 | | |
| 8 | 192-196 | 189-193 | | 180 | | |

**VIII-1** Explain how the complex number x + yi can be written in trigonometric form.

1.



The complex number x + yi is represented by the point P. Explain how you show that this number can be written in the form r(cos θ + i sin θ).

2. The trigonometric form of a complex number is r(cos θ + i sin θ). Explain how the complex number x + yi can be written in this trigonometric form.

3. Explain how the complex number x + yi can be written in its trigonometric form.

4. The trigonometric form of the complex number x + yi is ____________________.
Verify that a complex number can be written in these two forms.

**VIII-2** Plot complex numbers on the complex plane, given the numbers in rectangular or trigonometric form.

Plot and label each of the indicated sets of complex numbers on the complex plane.

1. a) $3 - 4i$

   b) $5(\cos 30° + i \sin 30°)$

   c) $4(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$

   d) $8(\cos 2.2 + i \sin 2.2)$

   e) $\frac{-5 + 4i}{2}$

2. a) $3(\cos 4.1 + i \sin 4.1)$

   b) $-6 + \frac{5}{2}i$

   c) $6(\cos 330° + i \sin 330°)$

   d) $5[\cos(-310°) - i \sin(-310°)]$

   e) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

3. a) $5 - 8i$

   b) $6.5 (\cos 4.3 + i \sin 4.3)$

   c) $5[\cos(-30°) + i \sin(-30°)]$

   d) $4 \cos 530° + 4i \sin 530°$

   e) $\frac{4 + 14i}{3}$

Pre-Calculus

Unit VIII

VIII-3 Convert complex numbers from the rectangular form to the trigonometric form and vice versa.

Express each of the following sets of complex numbers in its equivalent form (use tables, where necessary).

| | | Rectangular | Trigonometric |
|---|---|---|---|
| 1. | a) | $1 - i$ | ________ |
| | b) | ________ | $2(\cos 2\ 70° + i \sin 270°)$ |
| | c) | ________ | $15(\cos \frac{19}{6} \pi)$ |
| | d) | $3 + 4i$ | ________ |
| 2. | a) | ________ | $5(\cos \frac{7}{4} \pi + i \sin \frac{7}{4} \pi)$ |
| | b) | $-3\sqrt{3} - 3i$ | ________ |
| | c) | $2 - 3i$ | ________ |
| | d) | ________ | $6[(\cos(-214°) + i \sin(-214°)]$ |
| 3. | a) | $6 - 6\sqrt{3}i$ | ________ |
| | b) | $-4 - 2i$ | ________ |
| | c) | ________ | $\cos \frac{91}{4} \pi + i \sin \frac{91}{4} \pi$ |
| | d) | ________ | $2[\cos(5.8294) + i \sin(5.8294)]$ |

VIII-4 Use the trigonometric form of complex numbers to determine a product.

1. Let $z_1 = 3(\cos 80° + i \sin 80°)$ and $z_2 = 4(\cos 40° + i \sin 40°)$.

   Determine $z_1 \cdot z_2$ and express the product in trigonometric form.

2. Determine the product of

   $$[8(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)] \bullet [2(\cos \frac{7}{4}\pi + i \sin \frac{7}{4}\pi)]$$

   and leave the answer in trigonometric form.

3. Evaluate: $[2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})] \bullet [3(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)]$

   Leave the answer in trigonometric form.

4. Let $z_1 = 2(\cos 45° + i \sin 45°)$; $z_2 = \cos(-35°) + i \sin(-35°)$; and

   $z_3 = \frac{3}{2}(\cos 152° + i \sin 152°)$.

   Determine $z_1 \bullet z_2 \bullet z_3$ and express the answer in trigonometric form.

VIII-5 Use the trigonometric form of complex numbers to determine a quotient.

1. Determine the quotient of $\frac{4(\cos 145° + i \sin 145°)}{8(\cos 32° + i \sin 32°)}$.

   Leave the answer in trigonometric form.

2. Let $z_1 = 12[\cos(-10°) + i \sin(-10°)]$ and $z_2 = 4[\cos(56°) + i \sin(56°)]$.

   Evaluate $\frac{z_1}{z_2}$ and leave the answer in trigonometric form.

3. If $z_1 = 6(\cos 42° + i \sin 42°)$ and $z_1 \bullet z_2 = 24(\cos 193° + i \sin 193°)$,

   then $z_2 =$ _____. Leave the answer in trigonometric form.

4. Evaluate $\frac{4.8(\cos 38° + i \sin 38°)}{.4(\cos 42° - i \sin 42°)}$. Leave the answer in trigonometric form.

Pre-Calculus

Unit VIII

VIII-6 Apply DeMoivre's Theorem to determine a power of a complex number.

1. Let $z = 2(\cos 15° + i \sin 15°)$. Determine $z^7$.

2. Determine $z^{10}$ if $z = \cos(.4567) + i \sin(.4567)$.

3. Determine $z^{20}$ if $z = 1 - \sqrt{3}\, i$. Leave the answer in $x + yi$ form.

4. Apply DeMoivre's Theorem to determine $\left(\dfrac{6i}{\sqrt{2} + i\sqrt{2}}\right)^5$.

   Leave answer in trigonometric form.

VIII-7 Apply DeMoivre's Theorem to determine the roots of a complex number.

1. Using DeMoivre's Theorem, find the three cube roots of 8.

2. Apply DeMoivre's Theorem to determine the five fifth roots of $1 - \sqrt{3}\, i$. Express the answers in the trigonometric form.

3. Find all the cube roots of $-64\, i$. Express the answers in both the trigonometric and rectangular form.

4. If $z = 2(\cos 10° + \sin 10°)$ is one of the sixth roots of a complex number, determine the other five roots of this complex number.

Pre-Calculus

Unit VIII

VIII-8 Apply DeMoivre's Theorem to determine all the roots of an equation in the form $x^n = k$; k is a constant.

1. Apply DeMoivre's Theorem to determine all the roots of $x^3 = -27$.

   Leave the answers in rectangular form.

2. Apply DeMoivre's Theorem to determine all the roots if $x^5 - i = 0$.

   Leave the answers in trigonometric form.

3. Use DeMoivre's Theorem to determine the two roots of $x^2 + \sqrt{3} - i = 0$.

4. Solve for all roots of $z^3 = (1 + i)^2$. (HINT: Apply DeMoivre's Theorem.)

Pre-Calculus

Unit VIII

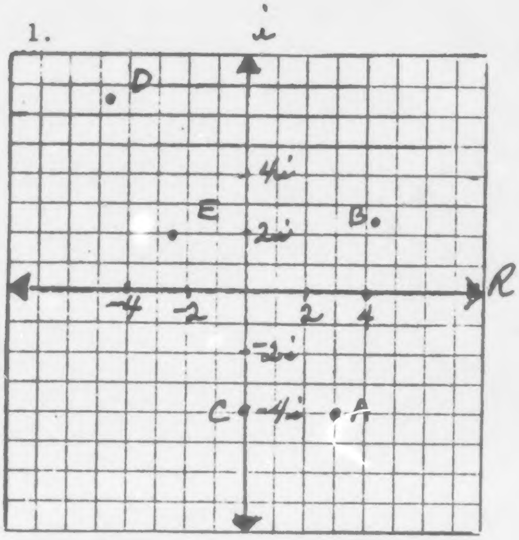
ANSWERS

VIII-1

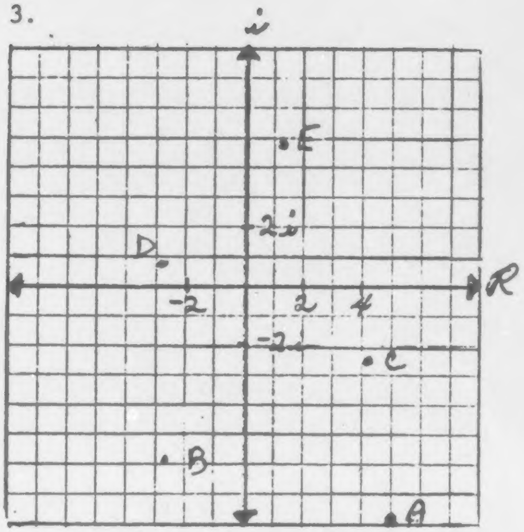
Refer to textbooks for acceptable answers.

VIII-2

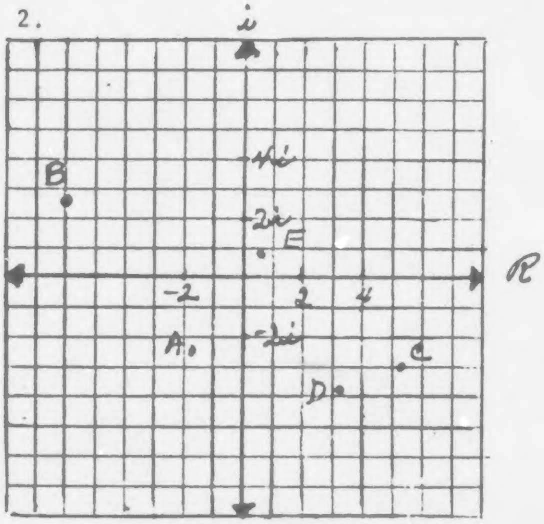
1.



3.



2.



Pre-Calculus

Unit VIII

ANSWERS

## VIII-3

1. a) $\sqrt{2}[\cos(-45^\circ) + i \sin(-45^\circ)]$

   b) $-2\,i$

   c) $\frac{-15}{2}\sqrt{3} - \frac{15}{2}\,i$

   d) $5[\cos(\text{Arc tan } \frac{4}{3}) + i \sin(\text{Arctan } \frac{4}{3})]$ or $5(\cos 53^\circ 10' + i \sin 53^\circ 10')$

2. a) $\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}\,i$

   b) $6(\cos 210^\circ + i \sin 210^\circ)$

   c) $\sqrt{13}[\cos(\text{Arc tan } \frac{-3}{2}) + i \sin(\text{Arc tan } \frac{-3}{2})]$ or

   $\sqrt{13}[\cos(-56^\circ 20') + i \sin(-56^\circ 20')]$

   d) $-4.9740 + 3.3552\,i$

3. a) $12[\cos(-60^\circ) + i \sin(-60^\circ)]$

   b) $2\sqrt{5}(\cos 206^\circ 34' + i \sin 206^\circ 34')$

   c) $\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$

   d) $1.7976 - .8767\,i$

## VIII-4

1. $12(\cos 120^\circ + i \sin 120^\circ)$

2. $16(\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2})$

3. $6(\cos \frac{13}{12}\pi + i \sin \frac{13}{12}\pi)$

4. $3(\cos 162^\circ + i \sin 162^\circ)$

## VIII-5

1. $\frac{1}{2}(\cos 113^\circ + i \sin 113^\circ)$

2. $3[\cos(-66^\circ) + i \sin(-66^\circ)]$

3. $4(\cos 151^\circ + i \sin 151^\circ)$

4. $12(\cos 80^\circ + i \sin 80^\circ)$

Pre-Calculus

Unit VIII

## ANSWERS

### VIII-6

1. $128(\cos 105^\circ + i \sin 105^\circ)$

2. $\cos(4.567) + i \sin(4.567)$

3. $-2^{19}(1 + \sqrt{3}\, i)$

4. $243(\cos 225^\circ + i \sin 225^\circ)$

### VIII-7

1. $2(\cos 0^\circ + i \sin 0^\circ)$ or $2$

   $2(\cos 120^\circ + i \sin 120^\circ)$ or $-1 + i\sqrt{3}$

   $2(\cos 240^\circ + i \sin 240^\circ)$ or $-1 - i\sqrt{3}$

2. $\sqrt[5]{2}\,(\cos 60^\circ + i \sin 60^\circ)$

   $\sqrt[5]{2}\,(\cos 132^\circ + i \sin 132^\circ)$

   $\sqrt[5]{2}\,(\cos 204^\circ + i \sin 204^\circ)$

   $\sqrt[5]{2}\,(\cos 276^\circ + i \sin 276^\circ)$

   $\sqrt[5]{2}\,(\cos 348^\circ + i \sin 348^\circ)$

3. $4\,(\cos 90^\circ + i \sin 90^\circ)$ or $4i$

   $4\,(\cos 210^\circ + i \sin 210^\circ)$ or $-2\sqrt{3} - 2i$

   $4\,(\cos 330^\circ + i \sin 330^\circ)$ or $2\sqrt{3} - 2i$

4. $2\,(\cos 70^\circ + i \sin 70^\circ)$

   $2\,(\cos 130^\circ + i \sin 130^\circ)$

   $2 \cos 190^\circ + i \sin 190^\circ)$

   $2\,(\cos 250^\circ + i \sin 250^\circ)$

   $2\,(\cos 310^\circ + i \sin 310^\circ)$

Pre-Calculus

Unit VIII

ANSWERS

VIII-8

1. $\frac{3}{2} + \frac{3\sqrt{3}}{2} i$

   $-3$

   $\frac{3}{2} - \frac{3\sqrt{3}}{2} i$

2. $x_1 = \cos 18° + i \sin 18°$ or $\cos 378° + i \sin 18°$

   $x_2 = \cos 90° + i \sin 90°$

   $x_3 = \cos 162° + i \sin 162°$

   $x_4 = \cos 234° + i \sin 234°$

   $x_5 = \cos 306° + i \sin 306°$

3. $x_1 = \sqrt{2}(\cos 75° + i \sin 75°)$

   $x_2 \quad \sqrt{2}(\cos 255° + i \sin 255°)$

4. $x_1 = \sqrt[3]{2} (\cos 30° + i \sin 30°)$

   $x_2 = \sqrt[3]{2}(\cos 150° + i \sin 150°)$

   $x_3 = \sqrt[3]{2}(\cos 270° + i \sin 270°)$

## Unit IX Functions on the Natural Numbers

### Overview

An introduction to summations and the limit concept may be successfully accomplished through the study of sequences and series. Since the limit concept is important for many ideas of calculus, it is included in the material taught in a Pre-Calculus course. Mathematical induction is also presented as another method of proof. The expansion of a binomial is developed and applied to determine the value of a numerical expression.

### Suggestions to the Teacher

Since some of the topics in this unit may have been covered previously (refer to Unit XIII, Algebra II Course of Study), the background of the student should be carefully analyzed. If the student's background is weak, the presentation of sequences and series in an Algebra 2 textbook could be beneficial to the student. Otherwise, some of the objectives may be combined or even eliminated. The derivations of the formulas for the sum of finite arithmetic or geometric series as well as the summation notation should be stressed.

A consideration of infinite sequences motivates the introduction of the limit concept. The presentation of the limit concept is rather informal. The use of calculators may assist in the development of an intuitive understanding of limits.

The discussion of sequences and series leads naturally into the Principle of Mathematical Induction, which is used to establish statements involving summation. Mathematical induction is not to be confused with inductive arguments in science; rather it is a deductive process. The degree of precision in these proofs is left to the individual teacher.

The binomial theorem may be presented in several ways. Combinations and permutations provide one way, if time allows for the teaching of these concepts. Another approach may involve Pascal's triangle. In addition, an algorithm for writing the entire expansion may be developed step by step. Applications for the binomial expansion may be seen in the evaluation of numerical expressions, objective 27.

Suggested Time: 10 days

Pre-Calculus

## Unit IX Functions on the Natural Numbers

### PERFORMANCE OBJECTIVES

1. State the definition of a sequence.
2. Determine the first k terms of a sequence, given the formula for the nth term.
3. Determine the formula for the nth term of a given sequence.
4. Determine the first n terms of an arithmetic sequence, given the first term and the common difference.
5. Construct the first k terms of a harmonic sequence given either the first n terms of an arithmetic sequence or the nth term of the arithmetic sequence.
6. Apply the formula $a_n = a_1 + (n - 1)d$ to determine a specific term of a given arithmetic sequence.
7. Determine one or more arithmetic means between two given terms of an arithmetic sequence.
8. Given a series in summation notation, write it in expanded form.
9. Show the derivation of the formula for the sum of a finite arithmetic series.
10. Determine the sum of a finite arithmetic series using one or more formulas.
11. Given the values for some of the variables from the arithmetic sequence and series formulas, determine the value of a selected variable.
12. Determine the first k terms of a geometric sequence, given the first term and the common ratio.
13. Apply the formula $a_n = ar^{n-1}$ to determine the specific terms of a given geometric sequence.
14. Determine one or more geometric means between two given terms of a geometric sequence.
15. Show a derivation of the formula for the sum of a finite geometric series.
16. Determine the sum of a finite geometric series using one or more formulas.
17. Given values for some of the variables from the geometric sequence and series formulas, determine the value of a selected variable or variables.

## Unit IX Functions on the Natural Numbers

18. Given a set of sequences or series, classify them as being either arithmetic, geometric, or neither.

19. Given a series in expanded form, write it in summation notation.

20. Determine the sum of an infinite geometric series where $|r| < 1$.

21. State a definition for the limit of a sequence (of partial sums).

22. Determine the limit of a sequence and specify which terms of the sequence are contained in a given neighborhood of the limit.

23. State the Principle of Mathematical Induction.

24. Apply the Principle of Mathematical Induction to prove a given statement is true for all natural numbers.

25. Apply the binomial theorem to expand expressions of the form $(a + b)^n$.

26. Determine the rth term in the expansion of $(a + b)^n$.

27. Apply the binomial theorem to approximate the value of numerical expressions in the form $(1 + x)^p$, where $p \in$ rational numbers.

Unit IX - Functions on the Natural Numbers

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 211 | 391 | | 199 | | |
| 2 | 211-214 | 392-393 | | 200, 203 | | |
| 3 | 212-214 | 393-394 | | 203 | | |
| 4 | 214 | 400 | | 201, 203 | | |
| 5 | 242 | | | 200 | | |
| 6 | 214 | 401 | | 203 | | |
| 7 | | 400-401 | | | | |
| 8 | 244, 247 | 395-396 | | 201, 203 | | |
| 9 | 246 | 399 | | 201 | | |
| 10 | | 400-401 | | 202-203 | | |
| 11 | | 401 | | 203 | | |
| 12 | 214 | 403 | | 200, 203 | | |
| 13 | 214 | 403 | | 203 | | |
| 14 | | 403 | | | | |
| 15 | 245 | 402 | | 202 | | |
| 16 | | 402-405 | | 202-203 | | |

Unit IX - Functions on the Natural Numbers

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 17 |  | 403 |  | 203 |  |  |
| 18 | 246 |  |  | 203 |  |  |
| 19 | 246-247 |  |  |  |  |  |
| 20 | 247 | 421-422 |  | 205-206 |  |  |
| 21 | 228, 243 | 414 |  | 205 |  |  |
| 22 | 222-236 | 411-419 |  |  |  |  |
| 23 | 21-22 | 407 |  | 207 |  |  |
| 24 | 22-25 | 407-410 |  | 208-210 |  |  |
| 25 | 756-757 | 340-342 |  | 217-221 |  |  |
| 26 | 760 |  |  | 220-221 |  |  |
| 27 |  |  |  | 222 |  |  |

IX-1 State the definition of a sequence.

1. The definition of a sequence is __________.
2. State the definition of a sequence.
3. Define a sequence.

IX-2 Determine the first k terms of a sequence, given the formula for the nth term.

1. Write the first four terms of the sequence defined by $a_n = \frac{n}{n + 1}$.
2. Write the first five terms cf the sequence defined by $a_n = \frac{(-1)^{n-1}}{2} \cdot (2n - 1)$.
3. Write the first four terms of the sequence defined by $a_n = \sin (n \cdot \frac{\pi}{2})$.

**IX-3 Determine the formula for the nth term of a given sequence.**

Determine the simple formula for the nth term of a sequence, given:

1. $a_1 = 3,\ a_2 = 5,\ a_3 = 7,\ a_4 = 9$

2. $a_1 = 1,\ a_2 = \frac{-2}{3},\ a_3 = \frac{4}{9},\ a_4 = -\frac{8}{27}$

3. $a_1 = 2,\ a_{n+1} = a_n - 3$

**IX-4 Determine the first n terms of an arithmetic sequence, given the first term and the common difference.**

Write the next three terms of the arithmetic sequence, given:

1. $a_1 = \frac{3}{4}$ and $d = \frac{1}{2}$

2. $a_1 = b - a$ and $d = -b$

3. $a_1 = 2x^2 - 3x + 2$ and $d = 2x - 1$

IX-5 Construct the first k terms of a harmonic sequence, given either the first n terms of an arithmetic sequence or the nth term of the arithmetic sequence.

1. Given the sequence 1, 4, 7, 10, ..., write the first four terms of the corresponding harmonic sequence.

2. Given the arithmetic sequence {nx}, write the first four terms of the corresponding sequence.

3. Given the sequence 1, $\frac{2}{5}$, $-\frac{1}{5}$, $-\frac{4}{5}$, ..., write the first four terms of the corresponding harmonic sequence.

IX-6 Apply the formula of $a_n = a_1 + (n - 1)d$ to determine a specific term of a given arithmetic sequence.

1. Determine the 30th term of the arithmetic sequence 2.5, 5.5, 8.5, 11.5, ....

2. Determine the 50th term of the arithmetic sequence $x^2 - 35x$, $x^2 - 32x$, $x^2 - 29x$, ...

3. Determine the 66th term of the arithmetic sequence defined by $a_1 = -8.5$ and $a_{n+1} = a_n + 1.5$,

**IX-7** Determine one or more arithmetic means between two given terms of an arithmetic sequence.

1. Insert two arithmetic means between -3.6 and 2.4.
2. Insert three arithmetic means between $x^2 - 4x$ and $-x^2 - 4x$.
3. Insert three arithmetic means between $m - 2n$ and $3m - 6n$.

**IX-8** Given a series in summation notation, write it in expanded form.

Write each of the following in expanded form:

1. $\sum_{i=1}^{6} (3i - 2)$
2. $\sum_{k=1}^{5} \frac{k-2}{k}$
3. $\sum_{i=0}^{4} a_i x^{4-i}$

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IX-9 Show the derivation of the formula for the sum of a finite arithmetic series.

1. Show the derivation of the formula for the sum of a finite arithmetic series.

IX-10 Determine the sum of a finite arithmetic series using one or more formulas.

1. Determine the sum of the arithmetic series:

$$20 + 17.5 + 15 + 12.5 + 10 + 7.5 + 5.$$

2. Determine the sum of:

$$\sum_{k=1}^{30} (40 - 3k).$$

3. A marble rolls down an inclined plane, traveling 0.15 m the first second. In each succeeding second it travels 0.24 m more than in the preceding one. How far does it roll in 9 seconds?

**IX-11** Given the values for some of the variables from the arithmetic sequence and series formulas, determine the value of a selected variable.

1. Determine the common difference and the eighth term of the arithmetic series whose first term is 10 and whose sum for eight terms is 108.

2. Given an arithmetic series with $d = 0.25$, $a_n = 12.5$, and $a_1 = 3.5$, determine $n$ and $S_n$.

3. Given an arithmetic series with $S_{20} = -240$ and $a_{20} = 7$, determine $a_1$ and $d$.

**IX-12** Determine the first k terms of a geometric sequence, given the first term and the common ratio.

1. Write the first four terms of the geometric sequence, given that $a_1 = 3$ and $r = \frac{1}{2}$.

2. Write the first five terms of the geometric sequence, given that $a_1 = 12$ and $r = -.01$.

3. Write the first five terms of the geometric sequence, given that $a_1 = 1$ and $r = x$.

**IX-13** Apply the formula $a_n = ar^{n-1}$ to determine the specific terms of a given geometric sequence.

1. Determine the common ratio of a geometric series whose first term is 36 and whose fifth term is $\frac{4}{9}$.

2. Determine $a_1$, for the geometric series with $r = \frac{1}{5}$ and $a_4 = \frac{4}{25}$.

3. Determine $a_5$ for the geometric series with $a_7 = 16$ and $a_1 = \frac{1}{4}$.

**IX-14** Determine one or more geometric means between two given terms of a geometric sequence.

1. Insert a geometric mean between $\frac{2}{3}$ and $\frac{3}{2}$.

2. Insert two geometric means between $-.01$ and 10.

3. Insert two geometric means between $x$ and $x^3$.

**IX-15** Show a derivation of the formula for the sum of a finite geometric series.

1. Show a derivation of the formula for the sum of a finite geometric series.

**IX-16** Determine the sum of a finite geometric series, using one or more formulas..

1. Determine the sum of the geometric series $\frac{3}{4} + 1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27}$ .

2. Determine the sum of $\sum_{k=1}^{4} (\frac{1}{5})^k$ using an appropriate formula.

3. Determine the sum of the geometric series $x^{\frac{1}{2}} + x + \ldots + x^4$.

**IX-17** Given values for some of the variables from the geometric sequence and series formulas, determine the value of a selected variable or variables.

1. Given a geometric series with $a_1 = 2$, $r = .2$, and $a_n = .016$, determine $n$ and $S_n$.

2. Given a geometric series with $a_1 = 2$ and $S_3 = 26$, determine all values of $r$ and $a_3$.

3. Given a geometric series with $a_3 = 100$ and $S_3 = 300$, determine $r$ and $a_1$.

**IX-18** Given a set of sequences or series, classify them as being either arithmetic, geometric, or neither.

1. Classify each of the following sequences as A (arithmetic), G (geometric), or N (neither).

   a) $2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots$

   b) $x, 2x, 3x, 4x, \ldots$

   c) $\frac{1}{2}, -1, 2, -4, \ldots$

   d) $\frac{1}{x}, \frac{1}{2x}, \frac{1}{3x}, \frac{1}{4x}, \ldots$

   e) $1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \ldots$

**IX-19 Given a series in expanded form, write it in summation notation.**

1. Write the following series in summation notation:

   $2 + 5 + 7 + 10 + 13$

2. Write the following series in summation notation:

   $100 + 20 + 4 + .8$

3. Write the following series in summation notation:

   $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$

**IX-20 Determine the sum of an infinite geometric series where $|r| < 1$.**

1. Determine the sum of the following infinite geometric series:

   $2 + \frac{6}{5} + \frac{18}{25} + \ldots.$

2. Determine the sum of the following infinite geometric series:

$$\sum_{k=1}^{\infty} 12.10^{-2k}$$

3. A side of a square is 8 inches. The midpoints of its sides are joined to form an inscribed square, and this process is continued forever. Determine the sum of all perimeters.

**IX-21** State a definition for the limit of a sequence (of partial sums).

1. What is a definition for the limit of a sequence?

2. Write a definition for the limit of a sequence.

3. Define the limit of a sequence.

**IX-22** Determine the limit of a sequence and specify which terms of the sequence are contained in a given neighborhood of the limit.

1. Given the sequence $\left\{\frac{2n - 1}{n + 2}\right\}$, determine the limit (L) of the sequence and state which terms of the sequence are contained in $\langle L - .01, L + .01\rangle$.

2. Given the sequence $\left\{\frac{6 - n}{3n}\right\}$, determine the limit (L) of the sequence and state which terms of the sequence are contained in $\langle L - .1, L + .1\rangle$.

3. Given the sequence $\left\{\frac{2n - 1}{3n}\right\}$, determine the limit (L) of the sequence and state which terms of the sequence are contained in $\langle .3, .6\rangle$.

4. Given the sequence $\left\{\frac{3n - 1}{n + 10}\right\}$, determine the limit (L) of the sequence and state, in terms of $\varepsilon$, which terms of the sequence are contained in $\langle L - \varepsilon, L + \varepsilon\rangle$.

**IX-23 State the Principle of Mathematical Induction.**

1. What is the Principle of Mathematical Induction?

2. Write the Principle of Mathematical Induction.

3. State the Principle of Mathematical Induction.

**IX-24 Apply the Principle of Mathematical Induction to prove a given statement is true for all natural numbers.**

1. Use mathematical induction to show that the statement is true for all natural numbers: $P_n: 1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$.

2. Use mathematical induction to show that the statement is true for all natural numbers: $P_n: 3^n \geq 1 + 2n$.

3. Use mathematical induction to show that the statement is true for all natural numbers: $P_n: n^4 + 2n^3 + n^2$ is divisible by 4.

4. Use mathematical induction to show that the statement is true for all natural numbers: $P_n: x^{2n} - 1$ is divisible by $(x + 1)$.

IX-25 Apply the binomial theorem to expand expressions of the form $(a + b)^n$.

1. Write the first four terms in the expansion of $(x + 2y)^7$.

2. Write the complete expansion of $\left(\frac{x^3}{2} - 4y^2\right)^4$.

3. Write and simplify the first four terms of the expansion of $\left(x^{\frac{1}{3}} - y^{-\frac{1}{3}}\right)^{10}$.

4. Apply the binomial theorem and DeMoivre's Theorem to show that:

   $\cos 3\theta = \cos^3\theta - 3\cos\theta \cdot \sin^2\theta$ and

   $\sin 3\theta = 3\sin\theta \cdot \cos^2\theta - \sin^3\theta$.

   (HINT: Expand $(\cos\theta + i\sin\theta)^3$ by binomial theorem and DeMoivre's Theorem.)

Pre-Calculus

Unit IX

IX-26 Determine the rth term in the expansion of $(a + b)^n$.

1. Determine the sixth term in the expansion of $(a - b)^{13}$.

2. What is the fifth term in the expansion of $(x - 2y)^7$?

3. Determine the middle term in the expansion of $(2x^2 + 3y)^8$.

4. What is the eighth term in the expansion of $(3x + y^{\frac{1}{2}})^{12}$?

IX-27 Apply the binomial theorem to approximate the value of numerical expressions in the form $(1 + x)^p$, where $p \in$ rational numbers.

1. Find the value of $(1.01)^{-4}$ correct to four significant figures.

2. Assuming the binomial theorem applies to the rational numbers, approximate the value of $\sqrt{1.04}$.

3. Apply the binomial theorem to approximate $\sqrt{15}$ to four significant figures.

   HINT: $\sqrt{15} = \sqrt{16 - 1} = (16 - 1)^{\frac{1}{2}} = [16(1 - \frac{1}{16})]^{\frac{1}{2}} = 4(1 - \frac{1}{16})^{\frac{1}{2}}$

4. Compute an approximate value of $\sqrt[3]{120}$ by application of the binomial theorem.

Pre-Calculus

Unit IX

## ANSWERS

### IX-1

See any approved text.

### IX-2

1. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

2. $\frac{1}{2}, \frac{-3}{2}, \frac{5}{2}, \frac{-7}{2}$

3. 1, 0, -1, 0

### IX-3

1. $a_n = 2n + 1$

2. $a_n - (-\frac{2}{3})^{n-1}$

3. $a_n = 5 - 3n$

### IX-4

1. $a_2 = \frac{5}{4}, a_3 = \frac{7}{4}, a_4 = \frac{9}{4}$

2. $a_2 = -a, a_3 = -a - b, a_4 = -a - 2b$

3. $a_2 = 2x^2 - x + 1, a_3 = 2x^2 + x, a_4 = 2x^2 + 3x - 1$

### IX-5

1. $a_1 = 1, a_2 = \frac{1}{4}, a_3 = \frac{1}{7}, a_4 = \frac{1}{10}$

2. $a_1 = \frac{1}{x}, a_2 = \frac{1}{2x}, a_3 = \frac{1}{3x}, a_4 = \frac{1}{4x}$

3. $1, \frac{5}{2}, -5, -\frac{5}{4}, \ldots$

Pre-Calculus

Unit IX

ANSWERS

## IX-6

1. $a_{30} = 89.5$

2. $a_{50} = x^2 + 112x$

3. $a_{66} = 89$

## IX-7

1. $-1.6, 0.4$

2. $\frac{1}{2}x^2 - 4x, -4x, -\frac{1}{2}x^2 - 4x$

3. $\frac{3}{2}m - 3n, 2m - 4n, \frac{5}{2}m - 5n$

## IX-8

1. $1 + 4 + 7 + 10 + 13 + 16$

2. $-1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5}$

3. $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$

## IX-9

See any approved text.

## IX-10

1. $S_7 = 87.5$

2. $S_{30} = -195$

3. $S_9 = 9.99$

Pre-Calculus

Unit IX

## ANSWERS

### IX-11

1. $d = 1$, $a_8 = 17$

2. $n = 37$, $S_{37} = 296$

3. $a_1 = -31$, $d = 2$

### IX-12

1. $a_1 = 3$, $a_2 = \frac{3}{2}$, $a_3 = \frac{3}{4}$, $a_4 = \frac{3}{8}$

2. $a_1 = 12$, $a_2 = -.12$, $a_3 = .0012$, $a_4 = -.000012$

3. $a_1 = 1$, $a_2 = x$, $a_3 = x^2$, $a_4 = x^3$, $a_5 = x^4$

### IX-13

1. $r = \pm \frac{1}{3}$ ($\pm \frac{1}{3}i$, too, if complex ratios permitted)

2. $a_1 = 20$

3. $a_5 = 4$

### IX-14

1. $\pm 1$

2. $.1, -1$

3. $x^{\frac{5}{3}}$, $x^{\frac{7}{3}}$

### IX-15

See any approved text

Pre-Calculus

Unit IX

ANSWERS

IX-16

1. $\frac{781}{108}$

2. $\frac{156}{625}$

3. $S_8 = \frac{x^{\frac{1}{2}} (1 - x^4)}{1 - x^{\frac{1}{2}}}$

IX-17

1. $n = 4$, $S_3 = 2.496$

2. $r = -4$, $a_3 = 32$ or $r = 3$, $a_3 = 18$

3. $r = 1$, $a = 100$ or $r = -\frac{1}{2}$, $a = 400$

IX-18

1. a) G

   b) A

   c) G

   d) N

   e) N

IX-19

1. $\sum_{k=1}^{5} (3k - 1)$

2. $\sum_{k=1}^{4} 100 (.2)^{k-1}$

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Pre-Calculus

Unit IX

ANSWERS

IX-19 (continued)

3. $\sum_{k=0}^{4} (-1)^k \frac{x^{2k}}{(2k)!}$

IX-20

1. 5

2. 12/99

3. $64 + 32\sqrt{2}$

IX-21

See any approved text.

IX-22

1. $L = 2$ and for $n > 498$, $a_n \in \langle 1.99, 2.01 \rangle$

2. $L = -\frac{1}{3}$ and for $n > 20$, $a_n \in \langle -\frac{13}{30}, -\frac{7}{30} \rangle$

3. $L = \frac{2}{3}$ and for $n \leq 4$, $a_n \in \langle .3, .6 \rangle$

4. $L = 3$ and for $n > \frac{31-10\varepsilon}{\varepsilon}$, $a_n \in \langle 3 - \varepsilon, 3 + \varepsilon \rangle$

IX-23

See any approved text.

IX-24

The format (and tricks) for answers may vary among teachers, so only some hints are given.

2. Multiply each side of $3^k \geq 1 + 2k$ by 3.

## ANSWERS

IX-24 (continued)

3. $(k + 1)^4 + 2(k + 1)^3 + (k + 1)^2 = (k^4 + 2k^3 + k^2) + 4(k^3 + 3k^2 + 3k + 1)$

4. $x^{2k + 2} - 1$ can be written as

$x^2(x^{2k} - 1) + x(x + 1) - (x + 1)$ or

$x^2(x^{2k} - 1) + (x - 1)(x + 1)$

IX-25

1. $x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + \ldots\ldots$

2. $\frac{x^{12}}{16} - 2x^9y^2 + 24x^6y^4 - 128x^3y^6 + 256y^8$

3. $x^{\frac{10}{3}} - 10x^3y^{-\frac{1}{3}} + 45x^{\frac{8}{3}}y^{-\frac{2}{3}} - 120x^{\frac{7}{3}}y^{-1} + \ldots$

4. a) by Binomial Theorem $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) - 3\cos\theta\sin^2\theta - i\sin^3\theta$

b) $(\cos\theta + i\sin\theta)^3 = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$

c) by DeMoivre's Theorem:

$(\cos\theta + i\sin\theta)^3 = (\cos 3\theta + i\sin 3\theta)$

d) $(\cos 3\theta + i\sin 3\theta) = (\cos^3\theta - 3\cos\theta \cdot \sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$

e) if $a + bi = x + yi$, then $a = x$ and $b = y$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta \cdot \sin^2\theta$

$\sin 3\theta = 3\sin\theta \cdot \cos^2\theta - \sin^3\theta$

Pre-Calculus

Unit IX

ANSWERS

IX-26

1. $-1287a^8b^5$

2. $560\ x^3y^4$

3. $90720\ x^8y^4$

4. $192456\ x^5y^{\frac{7}{2}}$

IX-27

1. .9610

2. 1.0198

3. 3.8730

4. 4.9324

## Unit X Introduction to Analytic Geometry

### Overview:

This unit provides the student with a review of several topics discussed in Algebra 2 such as the distance formula, slope, and various forms for the equations of a line. In addition, the student will be introduced to the concept of directed distance, direction angles, and parametric equations.

### Suggestions to the Teacher

It is recognized that several of the approved textbooks do not emphasize (or cover at all) direction angles, direction cosines and parametric equations.

Pre-Calculus Mathematics (Shanks) is the textbook which follows the stated objectives most closely.

The objectives in XVII may be incorporated into this unit (or at the appropriate time during any unit), or Unit XVII may be treated as a separate unit.

### Suggested Time

8 days

Pre-Calculus

## Unit X Introduction to Analytic Geometry

### PERFORMANCE OBJECTIVES

1. Determine the coordinates of the midpoint of a line segment or a point which divides a segment into a given ratio.

2. Define direction angles, direction cosines, and direction number.

3. Determine the direction angles and direction cosines when given:

   a) two points
   b) a set of parametric equations
   c) the equation written in standard or general form ($Ax + By + C = 0$)

4. Determine a set of parametric equations of a line when given:

   a) two points which belong to the line
   b) a point on the line and the direction cosines
   c) the general or standard equation of the line ($Ax + By + C = 0$)

5. Given a set of parametric equations of a line,

   a) determine the slope
   b) write the equation of the line
   c) graph the line

6. Determine if three or more points are collinear in SEVERAL WAYS.

7. Given the equation of two lines, determine if these lines are parallel, are the same line, or intersect. If the intersection is nonempty, determine the coordinates of the point of intersection.

8. Prove that for any two perpendicular lines (except for horizontal and vertical lines) the product of their slopes is -1.

9. Given the equations of two intersecting lines, determine the acute angle of intersection by applying the formulas

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \text{or} \quad \cos \theta = c_1 . c_1' + c_2 . c_2' .$$

10. Determine the distance from a point to a line.

(If you are using a book with vector approach, use in Unit XII.)

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Unit X - Introduction to Analytic Geometry

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 433-434 | 35-36 | 21-27 | 307-309 | | 307-312 |
| 2 | 429-438 | 250-251 | 281-284 | 270-274 | | 14-15<br>79 |
| 3 | 429-438 | 250-251 | 281-284 | 270-274 | | 47-52 |
| 4 | 437-440 | 42-45 | 278-279 | 270-274 | | 47-52 |
| 5 | 442-448 | 42-45 | | 272-275 | | 51-52 |
| 6 | | 37-39 | 10 | 277-279 | | 58-63 |
| 7 | | 45-48 | 51-53 | 283-287 | 63-68 | 58-63<br>97-109 |
| 8 | 455-459 | 46-48 | | 284-287 | 65-68 | 58-63 |
| 9 | 452-455 | | 16-21 | 284-287 | | 31-33 |
| 10 | 457-459 | 50-53 | 53-59 | 369-371 | 68 | 89-97 |
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**X-1 Determine the coordinates of the midpoint of a line segment or a point which divides a segment into a given ratio.**

1. **Given A(0,-5) and B(-5,13), determine the coordinates of the midpoint of $\overline{AB}$.**

2. **Determine the coordinates of C such that $\frac{AC}{AB} = \frac{1}{2}$ where C is between A and B and A(1,-4) and B(7,8).**

3. Given A(1,-1) and B(-5,7) and that C is between A and B, determine the coordinates of C such that $\frac{AC}{CB} = \frac{1}{4}$.

4. Given A(-3, 2) and B(4,7), if C is between A and B, determine C such that $\frac{AC}{AB} = \frac{1}{7}$.

X-2 Define direction angles, direction cosines, and direction number.

1. Define: direction angles.
2. State the definition of direction cosines.
3. Direction angles are defined as ____________.
4. If the direction cosines of a line are $\frac{1}{\sqrt{10}}$ and $\frac{-3}{\sqrt{10}}$, then what could a set of direction numbers for this line be?

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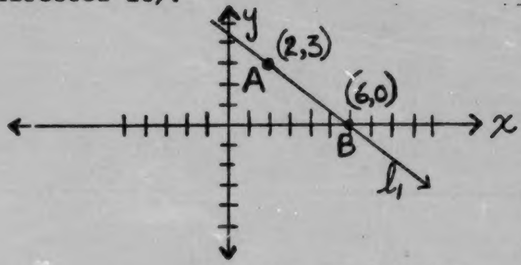
X-3 Determine the direction angles and direction cosines when given:

- a) two points
- b) a set of parametric equations
- c) the equation written in standard or general form $(Ax + By + C = 0)$.

1. Given A(4,5) and B(1,8), determine the direction cosines and the direction angles, $\alpha$ and $\beta$, for this line (as you have directed it).

2. Indicate the direction angles $\alpha$ and $\beta$ on the directed line, $\ell_1$, and determine $\alpha$ and $\beta$ to the nearest degree.



3. Given the parametric equation of $\ell_1$.

$$x = 2 - \frac{1}{\sqrt{10}} d$$

$$y = 3 - \frac{3}{\sqrt{10}} d ,$$

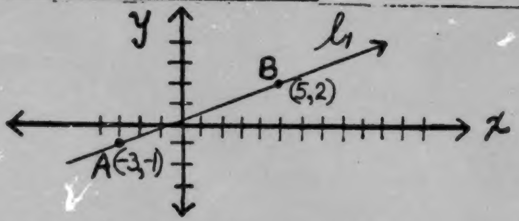
determine the direction angles, $\alpha$ and $\beta$ , to the nearest degree for this directed line.

4. Given the equation $x + 2y - 5 = 0$, determine a possible set of direction cosines and direction angles, $\alpha$ and $\beta$ , for this line.

**X-4** Determine a set of parametric equations of a line when given:

a) two points which belong to the line
b) a point on the line and the direction cosines
c) the general or standard equation of the line
(Ax = By + C = 0)

1. Given $\ell_1$ as directed, determine a set of parametric equations for this directed line.



2. Given A(0,-3) and B(-2,9), determine a set of parametric equations for this line.

3. A line contains the point (7,-2) and has $\cos \alpha = \frac{3}{5}$ and $\cos \beta = \frac{-4}{5}$. Determine a set of parametric equations for this line.

4. Determine a set of parametric equation for the line $2x - 3y + 6 = 0$.

**X-5** Given a set of parametric equations of a line,

a) determine the slope
b) write the equation of the line
c) graph the line

For each set of parametric equations,

a) determine the slope
b) write the equation of the line
c) graph the line

1. $$\begin{cases} x = 2 + \frac{1}{\sqrt{10}}\, d \\ y = 3 - \frac{3}{\sqrt{10}}\, d \end{cases}$$

2. $$\begin{cases} x = -2 + \frac{2}{\sqrt{5}}\, d \\ y = 3 - \frac{1}{\sqrt{5}}\, d \end{cases}$$

3. $$\begin{cases} x = 2 + \frac{3}{5}\, d \\ y = 4 - \frac{4}{5}\, d \end{cases}$$

4. $$\begin{cases} x = 4 \\ y = 2 + d \end{cases}$$

**X-6 Determine if three or more points are collinear in SEVERAL WAYS.**

1. Using slope, prove or disprove that the points A(0,-7), B(2,-1), and C(-5,-27) are collinear.

2. Using the definition of betweenness, prove or disprove that the points A(1,4), B(-3,12), and C(5,-4) are collinear.

3. Show in TWO DIFFERENT ways that the points A(1,-2), B(-3,10), and C(6,-17) are collinear.

4. Prove or disprove in TWO DIFFERENT ways that the points A(0,-2), B(2,4), and C(-4,-13) are collinear.

**X-7** **Given the equation of two lines. determine if these lines are parallel,** are the same line, or intersect. **If the intersection is nonempty, determine the coordinates of the point of intersection.**

**In each case, determine whether the two lines are parallel,** are the same line, or intersect. If they intersect, determine the point of intersection.

1. $\ell_1: \ 2x + 5y = 7$
   $\ell_2: \ y = -\frac{2}{5}x + 3$

2. $\ell_1: \begin{cases} x = 2 + \frac{1}{\sqrt{5}}\ d \\ y = -3 - \frac{2}{\sqrt{5}}\ d \end{cases}$

   $\ell_2: \begin{cases} x = -\frac{3}{5}\ d \\ y = 3 + \frac{4}{5}\ d \end{cases}$

3. $\ell_1: \ y = 2x + 7$

   $\ell_2: \begin{cases} x = -1 + \frac{2}{\sqrt{5}}\ d \\ y = 5 - \frac{1}{\sqrt{5}}\ d \end{cases}$

4. $\ell_1: \begin{cases} x = 2 + \frac{3}{5}\ d \\ y = -4 - \frac{4}{5}\ d \end{cases}$

   $\ell_2: \begin{cases} x = 5 - \frac{3}{5}\ d \\ y = -8 + \frac{4}{5}\ d \end{cases}$

**X-8** Prove that for any two perpendicular lines (except for horizontal and vertical lines) the product of their slopes is -1.

**Prove:** If $\ell_1 \perp \ell_2$ (except for horizontal and vertical lines), then $m_1 = -\frac{1}{m_2}$ $(m_1 m_2 = -1)$.

**X-9** Given the equations of two intersecting lines, determine the acute angle of intersection by applying the formulas

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \qquad \text{or } \cos\theta = c_1 \cdot c_1' + c_2 \cdot c_2' .$$

Given the equations of the following lines, determine the acute angle of intersection (to the nearest degree).

1. $\ell_1 : y = 4x - 5$
   $\ell_2 : 2x + y = 3$

2. $\ell_1 : 3x - 4y + 7 = 0$
   $\ell_2 : 2x + 3y + 8 = 0$

3. $\ell_1 : x = \frac{1}{4} y + 7$

   $\ell_2 : \begin{cases} x = 2 + \frac{1}{\sqrt{10}} d \\ y = -3 - \frac{3}{\sqrt{10}} d \end{cases}$

4. $\ell_1 : \begin{cases} x = 2 + d \\ y = -3 \end{cases}$

   $\ell_2 : \begin{cases} x = 4 + \frac{3}{\sqrt{13}} d \\ y = 6 + \frac{2}{\sqrt{13}} d \end{cases}$

X-10 Determine the distance from a point to a line.

1. Determine the distance from $(1,4)$ to the line $x + 2y - 3 = 0$.
2. $\Delta ABC$ is constructed where $A(0,4)$, $B(4,-3)$, and $C(-3,1)$. Determine the length of altitude to side AB.
3. Determine the distance between the parallel lines $4x - 3y + 12 = 0$ and $4x - 3y - 2 = 0$.
4. Which line is farther from the origin;

   $4x - 3y + 10 = 0$ or $5x - y - 15 = 0$?

**Pre-Calculus**

**Unit X**

## ANSWERS

### X-1

1. $(\frac{-5}{2}, 4)$
2. $(4, 2)$
3. $(\frac{-1}{5}, \frac{3}{5})$
4. $(-2, \frac{19}{7})$

### X-2

1. $\left.\begin{matrix}1.\\2.\\3.\end{matrix}\right\}$ Refer to textbook used.

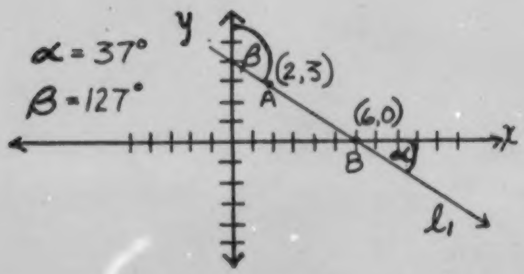
4. 1:-3 (or anything proportional)

### X-3

1. $$\begin{cases} \cos\alpha = \frac{-1}{\sqrt{2}} \\ \cos\beta = \frac{1}{\sqrt{2}} \\ \alpha = 135^\circ \\ \beta = 45^\circ \end{cases} \quad \text{or} \quad \begin{cases} \cos\alpha = \frac{1}{\sqrt{2}} \\ \cos\beta = \frac{-1}{\sqrt{2}} \\ = 45^\circ \\ = 135^\circ \end{cases}$$

2. $\alpha = 37^\circ$

   $\beta = 127^\circ$



## ANSWERS

### X-3 (continued)

3. $\alpha = 108^\circ$
   $\beta = 162^\circ$

4. $$\begin{cases} \cos\alpha = \dfrac{2}{\sqrt{5}} \\ \cos\beta = \dfrac{-1}{\sqrt{5}} \\ \alpha = 27^\circ \\ \beta = 117^\circ \end{cases} \quad \text{or} \quad \begin{cases} \cos\alpha = \dfrac{-2}{\sqrt{5}} \\ \cos\beta = \dfrac{1}{\sqrt{5}} \\ \alpha = 153^\circ \\ \beta = 63^\circ \end{cases}$$

### X-4

1. $$\begin{cases} x = -3 + \dfrac{8}{\sqrt{73}}\, d \\ y = -1 + \dfrac{3}{\sqrt{73}}\, d \end{cases}$$

   Answers vary depending on the point chosen for the origin.

2. $$\begin{cases} x = \dfrac{-1}{\sqrt{37}}\, d \\ y = -3 + \dfrac{6}{\sqrt{37}}\, d \end{cases}$$

   Answers vary.

3. $$\begin{cases} x = 7 + \dfrac{3}{5}\, d \\ y = -2 - \dfrac{4}{5}\, d \end{cases}$$

   Answers may vary.

4. $$\begin{cases} x = -3 + \dfrac{3}{\sqrt{13}}\, d \\ y = \dfrac{+2}{\sqrt{13}}\, d \end{cases}$$

   Answers may vary.

Pre-Calculus

Unit X

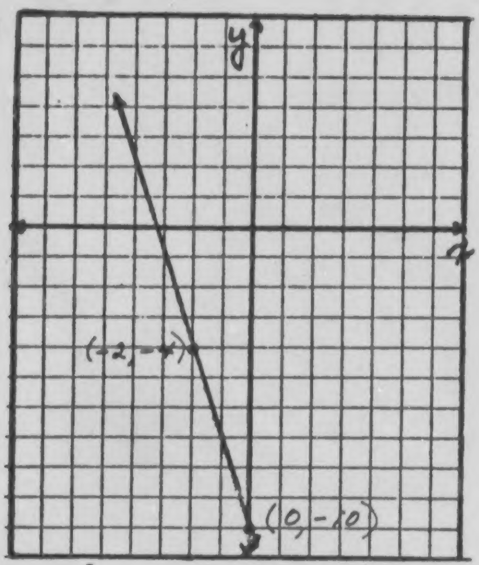
ANSWERS

X-5

1. a) $m = -3$

   b) $3x + y - 10 = 0$

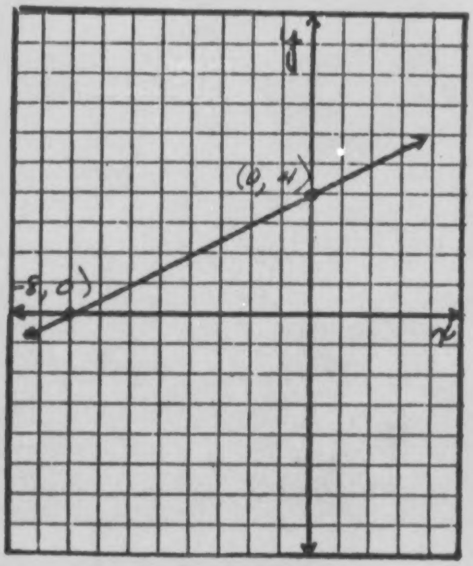
   c)



2. a) $m = \frac{1}{2}$

   b) $x - 2y + 8 = 0$

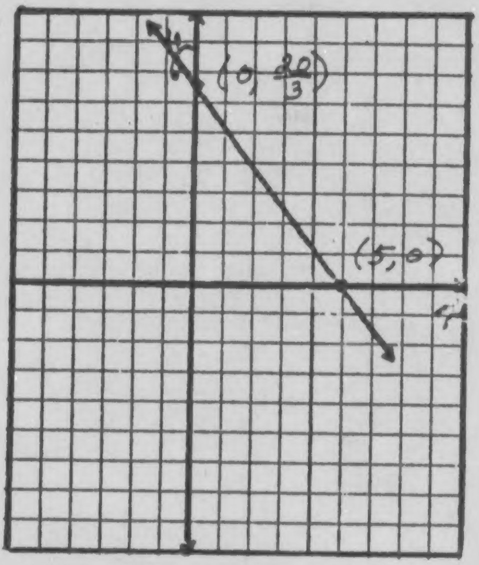
   c)



3. a) $m = -\frac{4}{3}$

   b) $4x + 3y - 20 = 0$

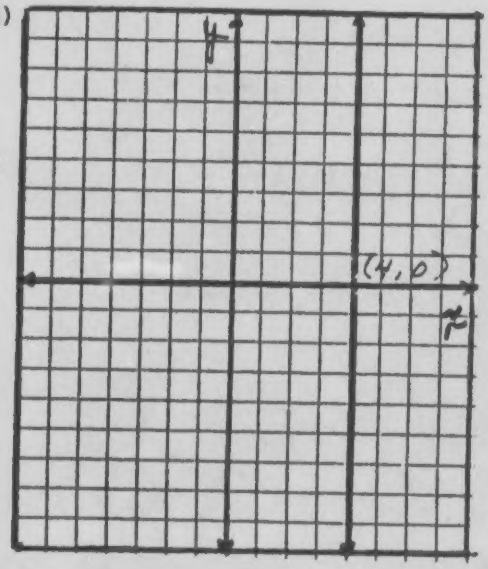
   c)



4. a) m is undefined

   b) $x = 4$

   c)



**Pre-Calculus**

**Unit X**

**ANSWERS**

X-6

1. $m_{AB} = 3$, $m_{BC} = \frac{26}{7}$, ($m_{AC} = 4$)

   Since $m_{AB} \neq m_{BC}$, the points are not collinear.

2. $AB = \sqrt{80} = 4\sqrt{5}$

   $BC = \sqrt{320} = 8\sqrt{5}$

   $AC = \sqrt{80} = 4\sqrt{5}$
   Since $AB + AC = BC$, the points are collinear.

3. Use slope or betweenness. Points are collinear.
   (It is also possible to use the equation of the line through two points and substitute the third point into the equation.

4. This is the same as #3. Points are not collinear.

X-7

1. parallel lines
2. intersecting lines (-3,7)
3. intersecting lines (-1,5)
4. same line

X-8

Refer to textbook used.

X-9

1. $\theta = 41^\circ$
2. $\theta = 71^\circ$
3. $\theta = 32^\circ$
4. $\theta = 34^\circ$

X-10

1. $\frac{6\sqrt{5}}{5}$
2. $\frac{33\sqrt{65}}{65}$
3. $\frac{14}{5}$
4. $5x - y - 15 = 0$

## Unit XI Points, Lines, and Planes in Space

### Overview

The approach to this unit will depend on teacher preference and the textbook used. The representation for a line in space will be new to the student. However, parametric equations and the symmetric form of a line are quite useful for the student.

### Suggestions to the Teacher

The labeling of the x, y, and z axes will depend on the textbook used. Any type of model will help the student visualize these points, lines, or planes being discussed. When determining the angle of intersection between two lines in space, the student should be encouraged to determine if the lines actually intersect. The geometric interpretation of solving linear systems of three equations in three unknowns might be useful for the student's understanding of when and why the intersection is a point, a line, or empty.

### Suggested Time

10 days.

Pre-Calculus

## Unit XI Points, Lines, and Planes in Space

PERFORMANCE OBJECTIVES

1. Plot points (x, y, z) in space.

2. Determine the distance between two points in space.

3. Determine the direction angles and direction cosines when given:

   a) two points
   b) a set of parametric equations

4. Determine a representation of a line in space (parametric equation or symmetric form) when given:

   a) two points
   b) a point and direction cosines
   c) two intersecting planes

5. Determine several additional points which belong to a given line and the coordinates of the points where the given line pierces the xy, xz, and/or yz coordinate planes.

6. Determine if three or more points are collinear.

7. Sketch a plane and determine the intercepts and traces, when given the equation of a plane written in the $Ax + By + Cz + D = 0$ form (not all A, B, C = 0).

8. Given the equations of two intersecting planes, sketch the planes, indicate the line of intersection and determine several points which belong to the line of intersection.

9. Given the equations of three planes, determine whether the intersection of these planes is a point, a line, or empty. If the intersection is nonempty, determine the point(s) of intersection.

10. Determine the acute angle between two lines in space by applying:

$$\cos \theta = c_1 \cdot c_1' + c_2 \cdot c_2' + c_3 \cdot c_3'$$

11. Determine the equation of a plane when given:

    a) three points
    b) the equation of the traces

Unit XI - Points, Lines, and Planes in Space

CROSS-REFERENCES

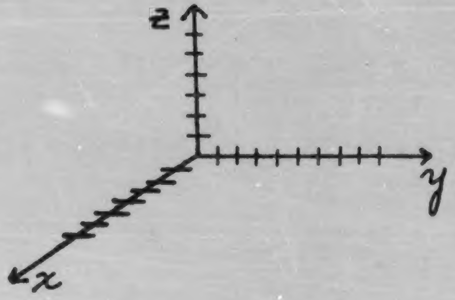
| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | | 242 | 227-228 | 314-317 | | 273-277 |
| 2 | 429 | 244-245 | 229-230 | 318-319 | | 276-277 |
| 3 | 429-432 | 250-251 | 281-283 | 320-323 | | 282-287 |
| 4 | 437-440 | 258-260 | 278-281 | 324-327 | | 307-311 |
| 5 | | 261 | | 325-326 | | 311-312 |
| 6 | | | | | | |
| 7 | 468 | | 235-236 | 327-328<br>331-332 | | 323 |
| 8 | 470-473 | | | 332 | | 324 |
| 9 | | | | 332 | | 311,<br>318-322 |
| 10 | 449-454 | 259-261 | 272-275 | 357-360 | | 311-312 |
| 11 | 465-469 | 262-264 | 273-277 | 329-332 | | 313-316 |

Unit XI

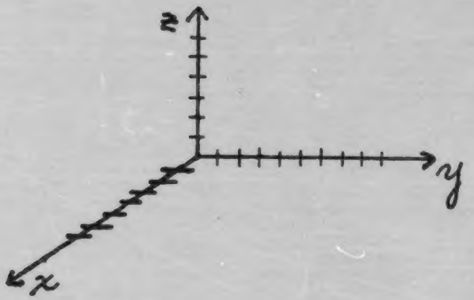
**XI-1 Plot points (x, y, z) in space.**

Plot and label these points on the x, y, z axes as shown.

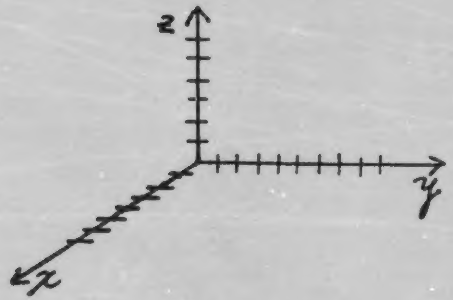
1. A(-1, 0, 5)
   B(5, 1, 4)
   C(1, 2, 4)
   D(-2, 3, -6)



2. A(4, 0, 3)
   B(-4, 2, 3)
   C(2, 3, 0)
   D(5, -4, -2)



3. A(0, 4, -3)
   B(-4, -8, 2)
   C(1, -2, 5)
   D(3, 3, -2)



**XI-2 Determine the distance between two points in space.**

Determine the distance between these points:

1. A(-1, 0, 4), B(2, -1, 5)
2. A(6, -2, 3), B(0, 2, 3)
3. A(-7, -5, 4), B(-5, 1, -3)
4. A$\left(\frac{7}{2}, \frac{3}{4}, 1\right)$, B(-1, 0, 4)

XI-3 Determine the direction angles and direction cosines when given:
a) two points
b) a set of parametric equations

1. Given the points A(4, 0, 2) and B(3, 2, 4), determine the direction cosines and the direction angles to the nearest **degree**.

2. Determine a set of direction cosines and direction angles (to the nearest degree) for the line containing (1, -3, 2) and (7, -6, 0).

3. Given $\ell_1$ represented by $\begin{cases} x = 2 + \frac{1}{9}d \\ y = 3 - \frac{4}{9}d \\ z = -4 - \frac{8}{9}d, \end{cases}$

   determine the direction angles to the nearest degree for this line.

4. Given $\ell_2$ represented by $\begin{cases} x = 2 \\ y = 1 - \frac{1}{\sqrt{2}}d \\ z = 4 + \frac{1}{\sqrt{2}}d, \end{cases}$

   determine $\alpha$, $\beta$, $\gamma$ (to the nearest degree).

XI-4 Determine a representation of a line in space (parametric equation or symmetric form) when given:

a) two points.
b) a point and direction cosines.
c) two intersecting planes

1. Determine a set of parametric equations for the line containing the points $A(-3, 0, 1)$ and $B(1, -1, 9)$.

2. Determine a set of parametric equation for the line containing the point $(1, -5, 6)$ and having

$$c_1 = \frac{2}{\sqrt{29}}, \quad c_2 = \frac{-3}{\sqrt{29}}, \quad \text{and } c_3 = \frac{-4}{\sqrt{29}}.$$

3. A line contains the points $(-1, 3, 4)$ and $(4, -6, 3)$. Draw and direct this line. According to the way you directed this line, determine a set of parametric equations for the line.

4. Given $\ell_1$ represented by $\begin{cases} 2x + z = 4 \\ 6x - y - 4z = 8, \end{cases}$

determine a set of parametric equations for this line.

**XI-5** Determine several additional points which belong to a given line and the coordinates of the points where the given line pierces the xy, xz, and/ or yz coordinate planes.

1. Given $\ell_1$ represented by $\begin{cases} x = 2 + \frac{1}{3}d \\ y = -1 - \frac{2}{3}d \\ z = 4 - \frac{2}{3}d \end{cases}$

   determine three points on this line.

2. Determine three points on the line represented by $\begin{cases} x = 2 + d \\ y = -3 \\ z = 4. \end{cases}$

3. Determine the coordinates of the point where the line $\begin{cases} x = 1 - \frac{3}{7}d \\ y = 2 + \frac{6}{7}d \\ z = -4 - \frac{2}{7}d \end{cases}$

   pierces the xy plane.

4. What are the coordinates of the points where the line $\begin{cases} x = -4 + \frac{8}{9}d \\ y = -\frac{4}{9}d \\ z = 3 + \frac{1}{9}d \end{cases}$

   pierces the xz and xy planes.

**XI-6 Determine if three or more points are collinear.**

Prove or disprove that the following points are collinear:

1. (11, 3, 0), (1, 1, 2), (-4, 0, 3)

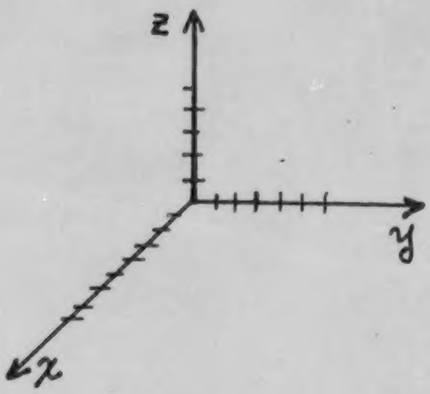
2. (2, -1, -3), (4, 0, 5), (6, 1, 12)

3. (0, 0, 2), (-3, 6, 4), (12, -23, -6)

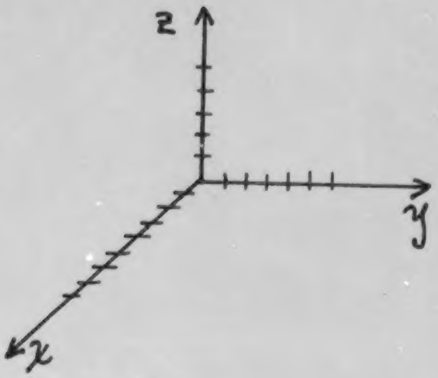
4. (4, 0, 0, ), (3, 0, 1), (12, 0, -8)

**XI-7** Sketch a plane, and determine the intercepts and traces, when given the equation of a plane written in the $Ax + By + Cz + D = 0$ form (not all $A, B, C = 0$).

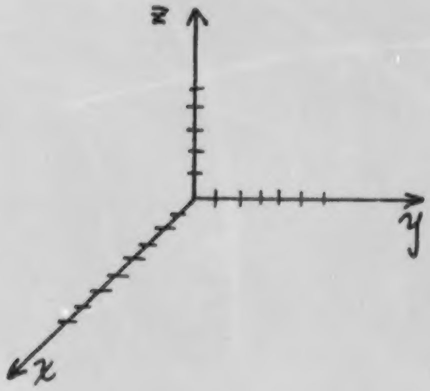
1. On the axes sketch the plane whose equation is $3x - y + 2z - 6 = 0$. Indicate the equations of the traces and the coordinates of the intercepts.



2. Sketch $3x + 4y = 12$.



3. Sketch $5x - 4y + 5z + 20 = 0$. Indicate the equations of the traces and the coordinates of the intercepts.



**XI-8** Given the equations of two intersecting planes, sketch the planes, indicate the line of intersection, and determine several points which belong to the line of intersection.

1. Sketch the planes whose equations are $\begin{cases} 2x - y + 2z - 10 = 0 \\ 4x + y + z - 8 = 0. \end{cases}$

   Show the line of intersection of these two planes.

2. Sketch the planes $\begin{cases} x + 4y = 8 \\ x + 4z = 8. \end{cases}$

   Show the line of intersection of these two planes.

3. After sketching the planes $\begin{cases} x + y + z = 4 \\ 2x - y - z = 8, \end{cases}$

   show the line of intersection and determine two points which belong to this line.

4. Sketch the planes $\begin{cases} x - y - z = 5 \\ 2x - y \phantom{- z} = 5. \end{cases}$

   Show the line of intersection and determine two points which belong to this line.

XI-9 Given the equations of three planes, determine whether the intersection of these planes is a point, a line, or empty. If the intersection is nonempty, determine the point(s) of intersection.

In each case determine whether the intersection of these planes is a point, a line, or empty. If it is a point, indicate the coordinates of the point. If it is a line, indicate at least two points on the line.

1. $3x + 2y - z = 4$

   $x + 3y + 2z = -1$

   $2x - y + 3z = 5$

2. $x + y + 4z = 1$

   $-2x - y + z = 2$

   $3x - 2y + 3z = 5$

3. $x - y + z = 3$

   $3x + y + z = 5$

   $-2x - z = -4$

4. $x - 3y + z = 3$

   $x + 2y - z = -2$

   $-3x - y + z = 0$

**XI-10** Determine the acute angle between two lines in space by applying:

$$\cos \theta = c_1 \cdot c_1' + c_2 \cdot c_2' + c_3 \cdot c_3'.$$

1. Given $\ell_1$ represented by $\begin{cases} x = 2 + \frac{1}{3} d \\ y = -1 - \frac{2}{3} d \\ z = \frac{2}{3} d \end{cases}$

   and $\ell_2$ represented by $\begin{cases} x = 2 \\ y = 1 - \frac{3}{5} d \\ y = -2 + \frac{4}{5} d, \end{cases}$

   determine to the nearest degree the angle between these lines.

2. Find the acute angle of intersection of the lines $\ell_1$ containing A(2, 3, 5) and B(6, -2, 2) and $\ell_2$ containing C(-2, -2, 8) and D(4, 1, 6).

3. Find the acute angle of intersection between the line $\overleftrightarrow{AB}$ and the line $\overleftrightarrow{CD}$ if A(5, 7, 3), B(-3, 3, 4), C(2, -3, 6), and D(3, -4, 1).

4. Find the acute angle between the line from the point (4, 5, -6) to the point (8, 6, 2) and a line having direction numbers 2: 4: -4.

XI-11 Determine the equation of a plane when given:

a) three points
b) the equation of the traces

1. Determine the equation of the plane containing the points (3, 1, 2), (4, -3, 5), and (1, 2, -6).

2. What is the equation of the plane which contains the points (2, 3, 0), (0, 4, 1), and (0, 0, -3).

3. Determine the equation of the plane having $x - 7y = 35$, $x - 5z = 35$, and $7y + 5z = -35$ as its traces.

4. Determine the equation of the plane which contains the line $\ell_1 \begin{cases} x = 2 - \frac{1}{3}d \\ y = -1 + \frac{2}{3}d \\ z = 4 - \frac{2}{3}d \end{cases}$

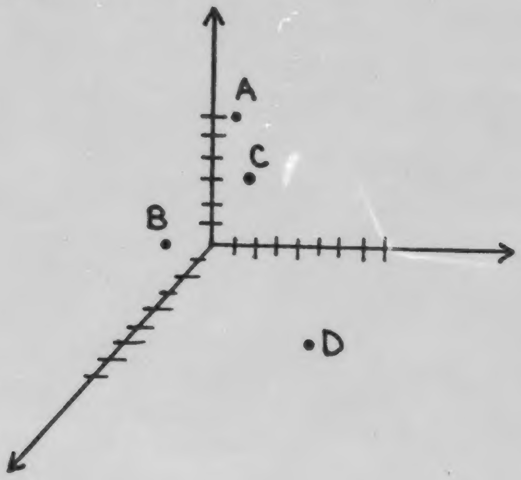
and the point (0, 1, 5).

**Pre-Calculus**

**Unit XI**

## ANSWERS

**XI-1.** (/ only shown)



**XI-2.**

1) $AB = \sqrt{11}$

2) $AB = 2\sqrt{13}$

3) $AB = \sqrt{89}$

4) $AB = \frac{\sqrt{477}}{4}$

**XI-3.**

1. $c_1 = \mp \frac{1}{3}$ $\alpha = 109^\circ$ or $71^\circ$

   $c_2 = \pm \frac{2}{3}$ $\beta = 48^\circ$ or $132^\circ$

   $c_3 = \pm \frac{2}{3}$ $\gamma = 48^\circ$ or $132^\circ$

2. $c_1 = \pm \frac{6}{7}$ $\alpha = 31^\circ$ or $149^\circ$

   $c_2 = \pm \frac{3}{7}$ $\beta = 65^\circ$ or $115^\circ$

   $c_3 = \mp \frac{2}{7}$ $\gamma = 107^\circ$ or $73^\circ$

3. $\alpha = 84^\circ$

   $\beta = 116^\circ$

   $\gamma = 153^\circ$

4. $\alpha = 90^\circ$

   $\beta = 135^\circ$

   $\gamma = 45^\circ$

Pre-Calculus

Unit XI

**ANSWERS**

XI-4. 1. $x = -3 \pm \frac{4}{9} d$

$y = 0 \mp \frac{1}{9} d$

$z = 1 \pm \frac{8}{9} d$

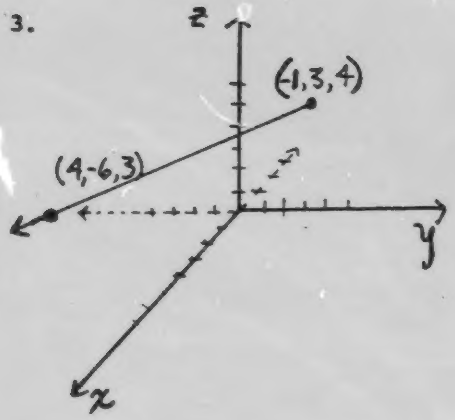
(Answers may vary.)

2. $x = 1 + \frac{2}{\sqrt{29}} d$

$y = -5 - \frac{3}{\sqrt{29}} d$

$z = 6 - \frac{4}{\sqrt{29}} d$

3.



$x = -1 + \frac{5}{\sqrt{107}} d$

$y = 3 - \frac{9}{\sqrt{107}} d$

$z = 4 - \frac{1}{\sqrt{107}} d$

Answers will vary.

4. $x = 2 - \frac{1}{\sqrt{201}} d$

$y = 4 - \frac{14}{\sqrt{201}} d$

$z = \frac{2}{\sqrt{201}} d$

(Answers will vary.)

XI-5. 1. Answers will vary.

2. Answers will vary.

3. (7, -10, 0)

4. (-4, 0, 3), (-28, 12, 0)

XI-6. 1. collinear

2. noncollinear

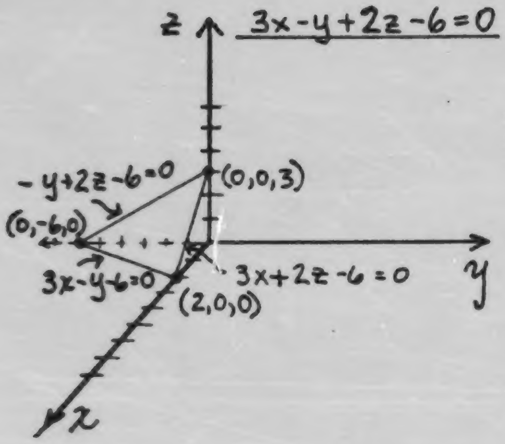
3. noncollinear

4. collinear

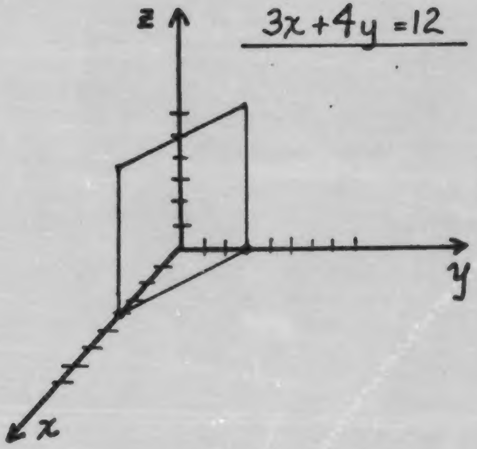
**ANSWERS**

**XI-7**

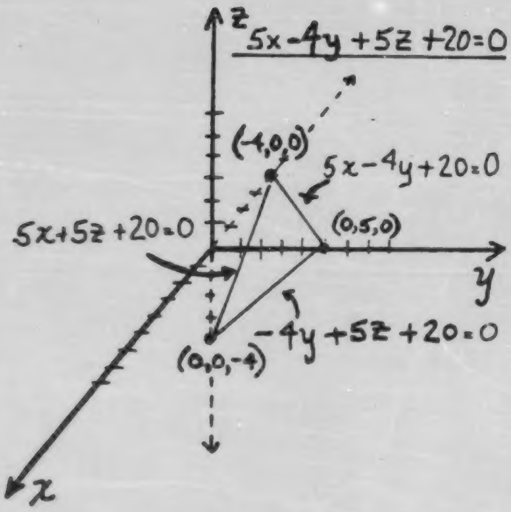
1.



2.



3.



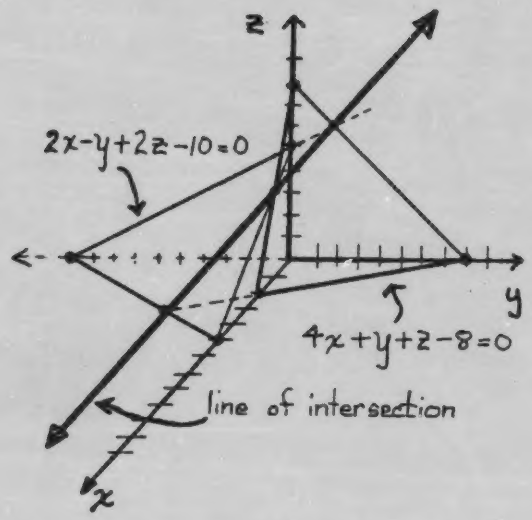
Pre-Calculus

Unit XI

**ANSWERS**

XI-8

1.



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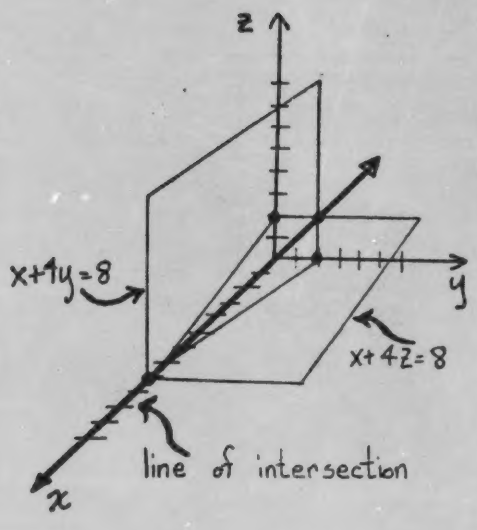

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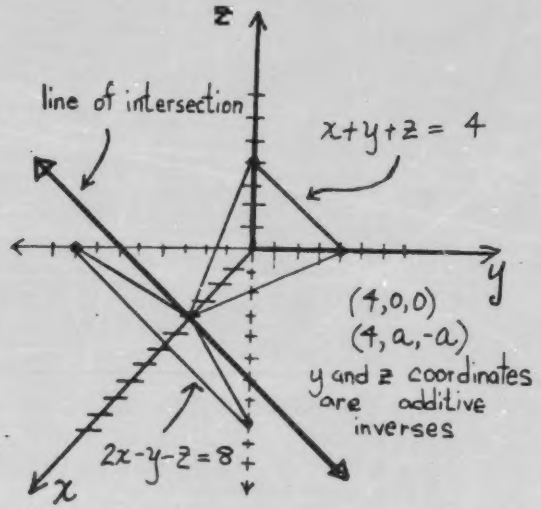

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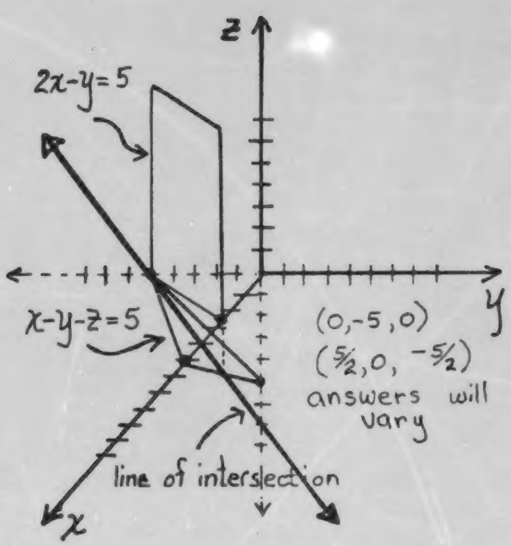


Pre-Calculus

Unit XI

## ANSWERS

### XI-9

1. (2, -1, 0)
2. $\left(\frac{1}{18}, \frac{-3}{2}, \frac{11}{18}\right)$
3. Intersection is a line.
   (1, 0, 2): (2, -1, 0): (3, -2, -2)
   Answers will vary.
4. $\emptyset$

### XI-10

1. 21°
2. 72°
3. 76°
4. 68°

### XI-11

1. $29x + 2y - 7z - 75 = 0$
2. $3x + 2y - 4z - 12 = 0$
3. $x - 7y - 5z - 35 = 0$
4. $6x + 5y + 2z - 15 = 0$

## Unit XII Vectors in a Plane

### Overview

This unit is an introduction to vectors and the operations of addition, scalar multiplication, and dot products. The formal structure of the unit is composed of definitions and a coordinate or component approach. An intuitive presentation of vectors is followed by geometric, physical, and coordinate-free definitions. Simplicity and clarity are stressed in the concepts of this unit.

### Suggestions to the Teacher

Notation may cause students difficulty at the start of this unit. Each approved textbook differs in its approach to vectors and the notation used. For example, the magnitude or norm of a vector may be represented by $\|\vec{a}\|$ or $|\vec{a}|$ while $\vec{v} = 2\vec{i} + 3\vec{j}$ may be written a $\vec{v} = \begin{pmatrix}2\\3\end{pmatrix}$. Because of these differences, it will be necessary for some of the assessment tasks to be changed to agree with the notation used by students. Emphasize the meaning and differences in such things as $\vec{u} \cdot \vec{v}$ and $r\vec{u}$. Several of the textbooks treat vectors in a plane and vectors in space as one unit. If desired, Units XII and XIII may be taught as one unit. Applications of vectors are included in the next unit.

### Suggested Time:

10 days

Pre-Calculus

## Unit XII Vectors in a Plane

### PERFORMANCE OBJECTIVES

1. Define a vector in terms of arrows in the plane or a directed line segment.
2. Given the coordinates of the initial point and terminal point of two vector representations, determine if the vectors are equivalent or determine the coordinates of a point such that the vectors are equivalent.
3. Given graphic representations of vectors, $\vec{a}$, $\vec{b}$, $\vec{c}$, ... apply the definitions of vector addition, vector subtraction, and scalar multiplication to determine the sum, difference, and/or scalar multiple of vectors.
4. Given the coordinates of the terminal points of position vectors, determine the terminal point of the vector which represents the sum, difference, and/or scalar multiple of these given vectors and/or write in the $a\vec{i} + b\vec{j}$ form.
5. Determine the magnitude (norm) of a vector represented by a given set of ordered pairs.
6. Given three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, each written in terms of its horizontal and vertical components, express one of the vectors as a linear combination of the other two vectors.
7. Determine and apply the dot product of two non-zero vectors.
8. Determine whether the vectors in a given pair are parallel, perpendicular, or neither parallel nor perpendicular. If they are neither parallel nor perpendicular, find the measure of the angle between them.
9. When given specific vectors, demonstrate the properties that relate to the operations of addition, scalar multiplication, and dot product.
10. Determine the vector equation of a line in a plane when given:
    a) two points
    b) the equation in the Ax + By + C = 0 form.

Unit XII - Vectors in a Plane

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 412 | 197-198 | 253 | 295 | | 5-6 |
| 2 | | 198-201 | | 292-294 | | 5-7 |
| 3 | 416-419 | 201-205 | 254-257 | 297-300 | | 8-13,<br>20-21 |
| 4 | 421 | 206-208,<br>220 | 259-263 | 299-300 | | 8-13,<br>20-21 |
| 5 | 412,428 | 211-213 | 259,264 | 295 | | 13-14 |
| 6 | 421-424 | 221-225 | | 301-303 | | |
| 7 | 450-453 | 226-232 | | 361-364 | | 26-30 |
| 8 | 451-455 | 226-232 | | 363-364 | | 16,<br>30-33 |
| 9 | 453 | 240 | | 342,364 | | 13,<br>22,28 |
| 10 | 437-440 | 258-261 | | 344-346 | | 47-51 |

XII-1 Define a vector in terms of arrows in the plane or a directed line segment.

1. Define a vector.
2. State the definition of a vector.
3. A vector is a ___?___.

XII-2 Given the coordinates of the initial point and terminal point of two vector representations, determine if the vectors are equivalent, or determine the coordinates of a point such that the vectors are equivalent.

1. Indicate by yes or no whether $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$ when given the specific coordinates for A, B, C, and D.

| | A | B | C | D |
|---|---|---|---|---|
| a) | (2, 4) | (4, -5) | (6, 3) | (8, -14) |
| b) | (0, 4) | (-1, 7) | (8, -2) | (7, -5) |
| c) | (-3, 4) | (2, -5) | (0, 0) | (5 -9) |

2. Point A is (3, -2) and B is (1, 3). For the given point C, find the point D such that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$.

a) C(-1, 5)

b) C(5, -7)

c) C(4, 4)

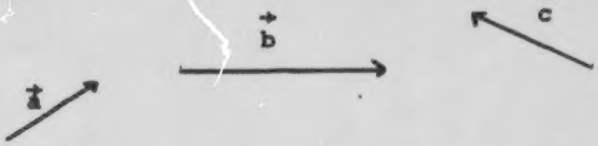
3. Determine the coordinates of the endpoint of the position vector, $\overrightarrow{OP}$, which is equivalent to $\overrightarrow{AB}$, given the coordinates of A and B.

a) A(6, 2) , B(-1, 5)

b) A(1, -3) , B(-6, -4)

c) A(-2, 0) , B(-4, 0)

XII-3 Given graphic representations of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, ..., apply the definitions of vector addition, vector subtraction, and scalar multiplication to determine the sum, difference, and/or scalar multiple of vectors.



Given $\vec{a}$, $\vec{b}$, and $\vec{c}$, show the representation for each of the following.

1. $\vec{a} + \vec{b}$

2. $\vec{a} + 2\vec{c}$

3. $\vec{b} - \frac{1}{2}\vec{c}$

4. $-(\vec{a} + \frac{1}{3}\vec{b}) + \vec{c}$

5. $-\vec{a} - (\vec{c} + 2\vec{b})$

XII-4 Given the coordinates of the terminal points of position vectors, determine the terminal point of the vector which represents the sum, difference, and/or scalar multiple of these given vectors and/or write in the $a\vec{i} + b\vec{j}$ form.

1. Given $\vec{r} = 3\vec{i} + 4\vec{j}$ and $\vec{s} = -\vec{i} + 5\vec{j}$. Determine $\vec{t}$ such that $\vec{t} = 2\vec{r} + 3\vec{s}$.

2. Given $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $u = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $s = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$. Write each of the following in the $a\vec{i} + b\vec{j}$ form.

   a) $2\vec{v} + \vec{u}$

   b) $3(5\vec{s} - 2\vec{v})$

   c) $\dfrac{\vec{v} + \vec{u} - 3\vec{s}}{2}$

3. A vector $\overrightarrow{PQ}$ is drawn from (1, 4) to (-7, 6). Represent the vector in in the $a\vec{i} + b\vec{j}$ form.

4. Let r = 3, s = -4, $\vec{v} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Determine each of the following:

   a) $r(\vec{v} + \vec{u})$

   b) $(r + s)(\vec{v} + 2\vec{u})$

   c) $r\vec{u} + s\vec{v}$

**XII-5** Determine the magnitude (norm) of a vector represented by a given set of ordered pairs.

Determine the magnitude (norm) if the position vectors are represented by the following ordered pairs; $\vec{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

1. $\| \vec{a} \|$

2. $\| \vec{a} + \vec{b} \|$

3. $\| \vec{a} - \vec{b} \|$

4. $\| 2\vec{c} + 3\vec{b} \|$

5. $\| -\vec{c} + 2\vec{a} \|$

XII-6 Given three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, each written in terms of its horizontal and vertical components, express one of the vectors as a linear combination of the other two vectors.

1. Given $\vec{a} = 3\vec{i} - 4\vec{j}$, $\vec{b} = 2\vec{i} + 3\vec{j}$, and $\vec{c} = \vec{i} + 10\vec{j}$, determine r and s such that $\vec{a} = r\vec{b} + s\vec{c}$.

2. Determine r and s such that:

$$5\vec{i} + 6\vec{j} = r(2\vec{i} - 3\vec{j}) + s\,(-6\vec{i} + 4\vec{j}).$$

3. Determine r and s so that $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ where $\vec{a} = \vec{i} + 7\vec{j}$, $\vec{b} = 2\vec{i} - \vec{j}$, and $\vec{c} = \vec{i} - 2\vec{j}$.

4. Using the vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, express $\vec{b}$ as a linear comination of $\vec{a}$ and $\vec{c}$, and express $\vec{c}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

XII-7 Determine and apply the dot product of two non-zero vectors.

1. Find each dot product if $\vec{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$, and $\vec{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

   a) $\vec{v} \cdot \vec{u}$

   b) $\vec{u} \cdot \vec{v}$

   c) $\vec{v} \cdot (\vec{u} + \vec{w})$

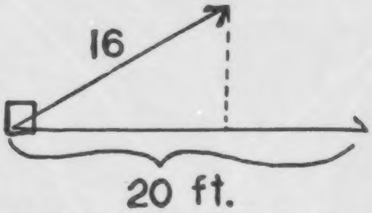
2. Find $\vec{u} \cdot \vec{v}$, if $\theta_1$ is the direction angle of $\vec{u}$ and $\theta_2$ is the direction angle of $\vec{v}$.

   a) $\|u\| = 3$ ; $\theta_1 = 30^\circ$, $\|v\| = 4$; $\theta_2 = 60^\circ$

   b) $\|u\| = 7$ ; $\theta_1 = 45^\circ$, $\|v\| = 5$; $\theta_2 = 135^\circ$

   c) $\|u\| = 9$ ; $\theta_1 = 60^\circ$, $\|v\| = 3$; $\theta_2 = 150^\circ$

3. A force F of 16 pounds is applied continuously at an angle of 30° to an object as shown. Compute the amount of work done by the force in moving the object 20 feet.



4. When a block of wood acted upon by a force of 48 pounds moves 10 feet in the direction of the force, the work done amounts to _____ foot pounds.

XII-8 Determine whether the vectors in a given pair are parallel, perpendicular, or neither parallel nor perpendicular. If they are neither parallel nor perpendicular, find the measure of the angle between them.

From each pair of vectors, determine if the vectors are parallel or perpendicular. If they are neither parallel nor perpendicular, determine the measure of the angle between them (to the nearest degree).

1. $\vec{a} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
2. $\vec{v} = \begin{pmatrix} 4 \\ 4\sqrt{3} \end{pmatrix}$, $\vec{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
3. $\vec{a} = 3\vec{i} + 4\vec{j}$, $\vec{b} = 4\vec{i} + 3\vec{j}$
4. $\vec{a} = 2\vec{i} + 3\vec{j}$, $\vec{b} = -4\vec{i} - 6\vec{j}$

XII-9 When given specific vectors, demonstrate the properties that relate to the operations of addition, scalar multiplication, and dot product.

Given $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $r = 2$, and $s = -3$, show that the following properties are true.

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $(r + s)\vec{c} = r\vec{c} + s\vec{c}$
3. $r(\vec{a} \cdot \vec{b}) = (r\vec{a}) \cdot \vec{b}$
4. $\vec{c} \cdot \vec{c} = \|c\|^2$

Pre-Calculus

Unit XII

XII-10 Determine the vector equation of a line in a plane when given:
a) two points
b) the equation in the $Ax + By + C = 0$ form.

1. Write a vector equation of a line containing $A(4, 1)$ and $B(-1, 3)$.

2. Determine the vector equation of a line containing $(2, -3)$ and $(4, -5)$.

3. Determine a vector equation of a line whose rectangular equation is $x + 2y = 5$.

4. Find a vector equation of a line whose rectangular equation is $5x - 3y = 10$.

Pre-Calculus

Unit XII

ANSWERS

**XII-1**

Refer to the textbook.

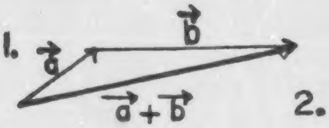
**XII-2**

1. a) no
   b) no
   c) yes

2. a) (-3, 10)
   b) ( 3, -2)
   c) ( 2, 9)

3. a) (-7, 3)
   b (-7, -1)
   c) (-2, 0)

**XII-3**

1.


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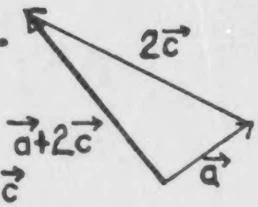

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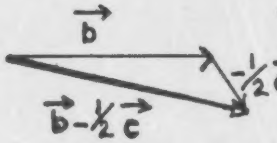

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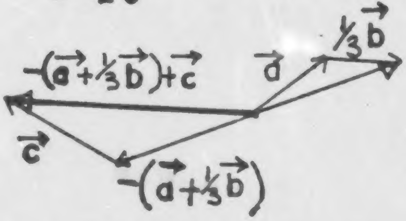
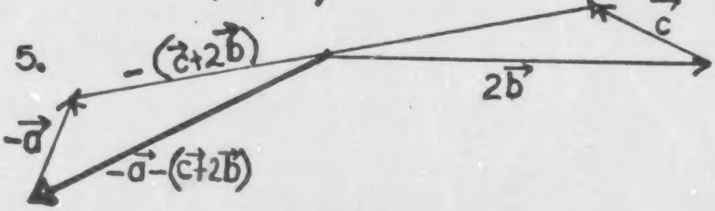


$-(\vec{a}+\frac{1}{3}\vec{b})+\vec{c}$
$\vec{a}$
$\frac{1}{3}\vec{b}$
$\vec{c}$
$-(\vec{a}+\frac{1}{3}\vec{b})$

5.


**XII-4**

1. $\vec{r} = 3\vec{i} + 19\vec{j}$

2. a) $6\vec{i} + 11\vec{j}$

   b) $-24\vec{i} + 57\vec{j}$

   c) $\vec{i} - \frac{7}{2}\vec{j}$

3. $-8\vec{i} + 2\vec{j}$

4. a) $21\vec{i} - 12\vec{j}$

   b) $-8\vec{i} + 7\vec{j}$

   c) $-21\vec{i} - 5\vec{j}$

**XII-5**

1. $\sqrt{13}$
2. $3\sqrt{10}$
3. $\sqrt{58}$
4. $\sqrt{481}$
5. $\sqrt{65}$

**XII-6**

1. $r = 2, s = -1$
2. $r = \frac{-28}{5}$, $s = \frac{-27}{10}$
3. $r = 3 \quad s = -5$
4. $\vec{b} = \frac{1}{3}\vec{a} + \frac{5}{3}\vec{c}$

   $\vec{c} = \frac{3}{5}\vec{b} - \frac{1}{5}\vec{a}$

**Pre-Calculus**

**Unit XII**

## ANSWERS

### XII-7

1. a) 18
   b) 18
   c) 17

2. a) $6\sqrt{3}$
   b) 0
   c) 0

3. $160\sqrt{3}$

4. 480

### XII-8

1. perpendicular
2. $120^{\circ}$
3. $16^{\circ}$
4. parallel

### XII-9

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$$\begin{array}{c|c} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ -1 \end{pmatrix} & \begin{pmatrix} 6 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix} \checkmark$$

2. $(r + s)\vec{c} = r\vec{c} + s\vec{c}$

$$\begin{array}{r|l} [2 + (-3)]\begin{pmatrix} 2 \\ 5 \end{pmatrix} & 2\begin{pmatrix} 2 \\ 5 \end{pmatrix} + (-3)\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ -1\begin{pmatrix} 2 \\ 5 \end{pmatrix} & \begin{pmatrix} 4 \\ 10 \end{pmatrix} + \begin{pmatrix} -6 \\ 15 \end{pmatrix} \\ \begin{pmatrix} -2 \\ -5 \end{pmatrix} = & \begin{pmatrix} -2 \\ -5 \end{pmatrix} \checkmark \end{array}$$

3. $r(\vec{a} \cdot \vec{b}) = (r\vec{a}) \cdot \vec{b}$

$$\begin{array}{r|l} 2(18 - 4) & \left[2\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right] \cdot \begin{pmatrix} 6 \\ -1 \end{pmatrix} \\ 2(14) & \begin{pmatrix} 12 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \end{pmatrix} \\ 28 & 32 - 8 \\ 28 = & 28 \checkmark \end{array}$$

4. $\vec{c} \cdot \vec{c} = \|c\|^2$

$$\begin{array}{r|l} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} & (\sqrt{29})^2 \\ 4 + 25 & \\ 29 = & 29 \checkmark \end{array}$$

### XII-10

1. $\vec{r} = (-1 + 5t)\vec{i} + (3 - 2t)\vec{j}$
   Answers may vary.

2. $\vec{r} = (2 + t)\vec{i} + (-3 - t)\vec{j}$
   Answers may vary.

3. $\vec{r} = t\vec{i} + (5 - 3t)\vec{j}$
   Answers may vary.

4. $\vec{r} = (2 + 3t)\vec{i} + (5t)\vec{j}$
   Answers may vary.

**Pre-Calculus** - Analytic Geometry

## Unit XIII Vectors in Space

### Overview

The structure of this unit is similar to that of Unit XII. The derivation of the equation of a plane is quite simple no matter whether one is using vectors or determining distances from a point to a line or plane. The unit deals with dot products and cross products and their applications.

### Suggestions to the Teacher

All of the objectives in Unit XII should be reviewed as they apply to vectors in space.

The concepts of a vector, the operations of addition and scalar multiplication, as well as the dot product should be easy for the student to extend into three dimensions. Applications of the dot product and cross product should be covered. The method of introducing distance from a point to a line or plane will differ slightly with the textbook used. After the idea of normals is discussed, parallel or perpendicular planes are easy for the student to understand.

### Suggested Time

10 days

## Unit XIII Vectors in Space

PERFORMANCE OBJECTIVES

1. Determine the angle between two vectors when given:
   a) the direction cosines of each vector
   b) the coordinates of A, B, and C of angle ABC
   c) the coordinates of the terminal point of two position vectors
2. Determine the dot product when given pairs of vectors.
3. Determine when two vectors are perpendicular or parallel.
4. Determine the distance from a point to a plane.
5. Determine the angle between planes.
6. Determine if two planes are parallel or perpendicular. If they are parallel, determine the distance between them.
7. Determine the cross product (vector product) when given two vectors.

Optional

8. Determine the equation of a plane when given:
   a) a unit normal and a point in the plane
   b) a unit normal and the distance of the plane from the origin
   c) three points
9. Apply cross product (vector product) to practical applications related to areas, moment of force, etc.

Unit XIII - Vectors In Space

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 449-454 | 223-251 | 272 | 357-359 |  | 31-32<br>287-292 |
| 2 | 450-453 | 227-232 | 267-268 | 361-364 |  | 26-29 |
| 3 | 60-61<br>426-428,451 | 227, 257 | - | 363-365 |  | 16, 20-21<br>26-27<br>282, 289 |
| 4 | 470-473 | 262-266 | 275-276 | 375-377 |  | 330-331 |
| 5 | 470-473 | 262-266 | 275 | 373-377 |  | - |
| 6 | - | 257 | - | 377 |  | 310, 315 |
| 7 | 461-465 | 253-255 | - | 378-380 |  | 294 |
| 8 | 465-468 | 262-263 | 273-274 | 384-386 |  | 313-318 |
| 9 | - | - | - | 384-386 |  | 297 |
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**Pre-Calculus**

**Unit XIII**

**XIII-1** Determine the angle between two vectors when given:

a) the direction cosines of each vector.
b) the coordinates of A, B, C of angle ABC.
c) the coordinates of the terminal point of two position vectors

1. The direction cosines of $\vec{a}$ are $\frac{1}{3}$, $\frac{-2}{3}$, and $\frac{-2}{3}$ and the direction cosines of $\vec{b}$ are $\frac{8}{9}$, $\frac{-4}{9}$, and $\frac{1}{9}$. Determine, to the nearest degree, the angle between $\vec{a}$ and $\vec{b}$.

2. Find, to the nearest degree, the angle between $\vec{a} = 3\vec{i} + 4\vec{j} - 5\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$.

3. Find, to the nearest degree, the angle between $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = 16\vec{i} + 8\vec{j} - 2\vec{k}$.

4. Determine the measure of angle A in $\triangle ABC$ if $A(0,4,5)$, $B(-1,2,-4)$, and $C(4,0,9)$.

Pre-Calculus

Unit XIII

**XIII-2 Determine the dot product when given pairs of vectors.**

1. If $\vec{a} = 8\vec{i} + \vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + 4\vec{k}$, find $\vec{a}\cdot\vec{b}$.

2. If $\vec{a} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$, find $\vec{a}\cdot\vec{b}$.

3. If $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = \vec{j} - 3\vec{k}$, and $\vec{c} = -2\vec{i} - \vec{j} + 4\vec{k}$, find $\vec{a}\cdot(\vec{b} + \vec{c})$.

4. Determine $\vec{a}\cdot\vec{b}$ if $\vec{a} = \vec{i}$ and $\vec{b} = -2\vec{i} + \vec{k}$.

**XIII-3** Determine when two vectors are perpendicular or parallel.

1. Determine if the following pairs of vectors are perpendicular or parallel.

   a) $\vec{a} = \vec{i} - 3\vec{j} + 5\vec{k}$

      $\vec{b} = 4\vec{i} - 2\vec{j} - 2\vec{k}$

   b) $\vec{a} = 2\vec{i} - 5\vec{j} + 6\vec{k}$

      $\vec{b} = \vec{i} - \frac{5\vec{j}}{2} + 3\vec{k}$

   c) $\vec{a} = 3\vec{i} - \vec{j} + 4\vec{k}$

      $\vec{b} = \vec{i} + 2\vec{j} + \frac{5\vec{k}}{4}$

2. Determine the value of x, y, or z so that the second vector is perpendicular to the first vector.

   a) $\vec{a} = 3\vec{i} - 6\vec{j} - 2\vec{k}$, $\vec{b} = 4\vec{i} - 2\vec{j} + z\vec{k}$

   b) $\vec{a} = 4\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = x\vec{i} + 3\vec{j} - 4\vec{k}$

   c) $\vec{a} = \vec{i} - 7\vec{j} + 6\vec{k}$, $\vec{b} = 7\vec{i} - y\vec{j} + 5\vec{k}$

3. If $a = 2\vec{i} - 4\vec{j} - 2\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + \vec{k}$, and $\vec{c} = \vec{i} - \vec{j} - 3\vec{k}$, which pairs, if any, are perpendicular?

4. If the vector $a\vec{i} + 2\vec{j} + \vec{k}$ is perpendicular to the vector $5\vec{i} - a\vec{j} + 7\vec{k}$, determine the value of a.

## XIII-4 **Determine the distance from a point to a plane.**

**Find the distance between the given plane and the given point.**

1. $x - 2y + 2z - 3 = 0$, $(1,1,1)$
2. $8x - y + 4z + 18 = 0$, $(-2,0,4)$
3. $6x - 2y - 3z + 2 = 0$, $(-1,4,1)$
4. $3x - 5z - 1 = 0$, $(3,7,2)$

## XIII-5 **Determine the angle between planes.**

**Determine the acute angle of intersection between the given planes.**

1. $x - 2y + 2z - 9 = 0$ and $x + 2y - 2z - 1 = 0$
2. $2x + 2y + z + 5 = 0$ and $5x + 4y - 3z - 2 = 0$
3. $2x - y - z + 8 = 0$ and $x + 2y + z - 4 = 0$
4. $x + 2y + 3z - 6 = 0$ and $2x + 5y - 4z - 10 = 0$

XIII-6 Determine if two planes are parallel or perpendicular. If they are parallel, determine the distance between them.

Determine which planes are parallel or perpendicular. If they are parallel, determine the distance between them.

1. $4x - 2y + 8z - 11 = 0$ and $6x - 3y + 12z + 13 = 0$
2. $3x - y + 2z = 6$ and $6x + 4y - 7z + 2 = 0$
3. $x + 2y + 3z - 6 = 0$ and $2x + 5y - 4z - 10 = 0$
4. $9x - 6y + 2z - 11 = 0$ and $18x - 12y + 4z - 55 = 0$

XIII-7 Determine the cross product (vector product) when given two vectors.

1. Let $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} + \vec{k}$. Determine $\vec{a} \times \vec{b}$.

2. If $\vec{t} = \vec{i} + 5\vec{j} + 8\vec{k}$ and $\vec{v} = 3\vec{i} + 2\vec{j} + 4\vec{k}$, determine the cross product $\vec{t} \times \vec{v}$.

3. Let $\vec{t} = \vec{i} + 5\vec{j} + 8\vec{k}$, $\vec{v} = 3\vec{i} + 2\vec{j} + 4\vec{k}$, and $\vec{s} = -\vec{i} + 2\vec{j} + 6\vec{k}$.
   a) Verify $\vec{t} \times \vec{v} = -(\vec{v} \times \vec{t})$.
   b) Verify $\vec{t} \times (\vec{v} + \vec{s}) = (\vec{t} \times \vec{v}) + (\vec{t} \times \vec{s})$.
   c) Verify $\vec{t} \times (2\vec{v}) = (2\vec{t}) \times \vec{v} = 2(\vec{t} \times \vec{v})$.
4. Determine a vector perpendicular to each of the vectors $2\vec{i} + \vec{j} + 5\vec{k}$ and $\vec{i} - 3\vec{j} - \vec{k}$.

XIII-8 **Determine the equation of a plane when given:**

a) a unit normal and a point in the plane.
b) a unit normal and the distance of the plane from the origin.
c) three points

1. Determine the equation of the plane containing the points A(1,3,-2), B(2,0,1), and C(0,-3,4).

2. Determine an equation of a plane through the point (-4,2,-5) and perpendicular to $\vec{a} = \vec{i} + 3\vec{j} + 2\vec{k}$.

3. Find an equation for the plane that contains (1,-3,2) and is perpendicular to the planes with equations $x + 2y + 3z = 4$ and $3x - 4y - 4z = 2$.

4. Find the equation of the plane which contains (4,6,2) and has a unit normal of $\vec{n} = \frac{3}{\sqrt{26}}\vec{i} - \frac{1}{\sqrt{26}}\vec{j} + \frac{4}{\sqrt{26}}\vec{k}$.

XIII-9 Apply cross product (vector product) to practical applications related to areas, moment of force, etc.

1. Using cross products, determine the area of $\Delta ABC$ if $A(2,-3,-1)$, $B(0,1,-2)$, and $C(1,2,3)$.

2. The volume of a tetrahedron can be expressed in terms of cross products as $v = \frac{1}{6}\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|$. Using this formula, determine the volume of the tetrahedron determined by origin and the points with position vectors $\vec{a} = 2\vec{i} - 3\vec{j} - \vec{k}$, $\vec{b} = \vec{j} - 2\vec{k}$, and $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$.

3. If $\vec{u} = 2\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$, find $\vec{u} \times \vec{v}$ and verify that it is perpendicular to both $\vec{u}$ and $\vec{v}$.

4. Using the formula: $\left\| \vec{n} \times \vec{n}^{1} \right\| = \left\| \vec{n} \right\| \cdot \left\| \vec{n}^{1} \right\| \cdot \sin \theta$ ($\vec{n}$ and $\vec{n}^{1}$ are the normal vectors to the plane), determine to the nearest degree the acute angle between $x + y + 2z = 7$ and $x - 2y - z = 4$.

**Pre-Calculus**

**Unit XIII**

## ANSWERS

### XIII-1

1. $59^\circ$
2. $30^\circ$
3. $75^\circ$
4. $120^\circ$

### XIII-2

1. 18
2. -9
3. 1
4. -2

### XIII-3

1. a. perpendicular
   b. parallel
   c. perpendicular

2. a. $z = 12$

   b. $x = \frac{3}{2}$

   c. $y = \frac{-37}{7}$

3. $\vec{b} \perp \vec{c}$

4. $a = \frac{-7}{3}$

### XIII-4

1. $\frac{2}{3}$

2. 2

3. $\frac{15}{7}$

4. $\frac{\sqrt{34}}{17}$

### XIII-5

1. $39^\circ$
2. $45^\circ$
3. $60^\circ$
4. $90^\circ$

### XIII-6

1. parallel, $\frac{59}{6\sqrt{21}}$
2. perpendicular
3. perpendicular

4. parallel, $\frac{3}{2}$

### XIII-7

1. $\vec{i} - 3\vec{j} - 7\vec{k}$

2. $4\vec{i} + 20\vec{j} - 13\vec{k}$

3. verify

4. $14\vec{i} + 7\vec{j} - 7\vec{k}$

### XIII-8

1. $y + z = 1$
2. $x + 3y + 2z + 8 = 0$
3. $4x + 13y - 10z + 55 = 0$
4. $3x - y + 4z - 14 = 0$

### XIII-9

1. $\frac{3}{2}\sqrt{62}$

2. $\frac{7}{2}$

3. $5\vec{i} - 3\vec{j} - 4\vec{k}$

4. $60^\circ$

## Unit XIV Conic Sections

### Overview

The study of second degree equations is expanded to include the conic sections with centers at the origin and (h,k). Degenerate conics are also presented at this time. In addition, translation and rotation of axes are considered in terms of the conic sections.

### Suggestions to the Teacher

The objectives in this unit allow the teacher some flexibility in planning class discussions. The development of the standard equation for each conic section may be approached either by using the definition of the conic section or by introducing eccentricity, e. Since eccentricity is a new concept, the teacher may choose to develop the equations by using this method. Objectives 8 and 12 may be considered together as one objective; however, the degenerate cases are allowed in both objectives. Translations and rotations of axes are also performed with the conic sections.

### Suggested Time

17 days

## Unit XIV Conic Sections

### PERFORMANCE OBJECTIVES

1. Determine the equation of a circle when given certain conditions.
2. Show the development of the standard equation of a parabola, either with center (0,0) or center (h,k).
3. Determine the equation of a parabola from the information given.
4. Show the development of the standard equation of an ellipse, either with center (0,0) or center (h,k).
5. Determine the equation of an ellipse from the information given.
6. Show the development of the standard equation of a hyperbola, either with center (0,0) or center (h,k).
7. Determine the equation of a hyperbola from the information given.
8. Given the equation of a conic section in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

    a) write the equation in standard form
    b) identify the conic, including the degenerate case
    c) determine the eccentricity
    d) discuss its properties, which may include its center, foci, radius, vertices, axis of symmetry, directrices, asymptotes, major and minor axes, transverse and conjugate axes
    e) graph the conic

9. Determine the point(s) of intersection, if any, of two conic sections.
10. Determine the new coordinates of a given point if the origin is moved by translation.
11. Determine the equation of a conic section by translating the origin to the point (h,k). Sketch its graph.
12. Given the equation of a conic section in the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

    identify the conic, including the degenerate case.

13. Determine the angle of rotation necessary to remove the xy-term in the general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Assume that the conic section is not the degenerate case.
14. Remove the xy-term from an equation of a conic section by a rotation of axes. Sketch the graph of the conic, showing both sets of axes.
15. Solve verbal problems involving conic sections.

## Unit XIV - Conic Sections

## CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 63-64 | 273-275 | 66-73 | 425-429 | | 125-129 |
| 2 | 607-608 | 275-277 | 83-92 | 442-445 | 76-80 | 130-136 |
| 3 | 602-605 | 275-277 | 87-89<br>91-92<br>95 | 442-445 | 76-80 | 130-136 |
| 4 | 583-592<br>608-609 | 277-280 | 97-105 | 446-451 | | 137-140<br>153-155 |
| 5 | 586-592<br>608-611 | 280-282 | 100<br>103-107 | 446-451 | | 137-144 |
| 6 | 593-601<br>609-610 | 282-283 | 108-112 | 452-458 | | 144-150 |
| 7 | 595-596<br>598-600<br>611 | 284-285 | 116-117 | 452-458 | | 150-153<br>158-160 |
| 8 | 582-611 | 286-289 | 82-118 | 462-464 | | 125-163 |
| 9 | - | - | - | 458 | | - |
| 10 | 612-616 | 109-112 | 77-80 | 475-476 | | 171 |
| 11 | 612-616 | 112 | 119-123 | 475-477 | | 171-174<br>177-178 |
| 12 | 612-616 | 286-289 | 119-123 | 477-481 | | 177-178 |
| 13 | 616-621 | 113-114<br>286-289 | 123-134 | - | | 180-191 |
| 14 | 616-621 | 113-114<br>286-289 | 126-134 | | | 180-191 |
| 15 | | 277, 282 | 106-107<br>114-117 | 461, 467 | | 136-144 |
| | | | | | | |

Pre-Calculus

Unit XIV

XIV-1 Determine the equation of a circle when given certain conditions.

1. Determine the equation of a circle with center $(-2, -1)$ and radius 4.

2. Determine the equation of a circle with center $(2, \pi)$ and radius $\sqrt{3}$ .

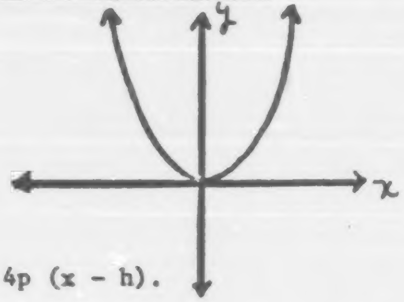
3. Determine the equation of the circle with center on the line $2x + 3y = 3$ and its tangent to both axes.

4. A circle has its center at the point $(-2, -5)$ and is tangent to the line $x + 2y = 8$. Determine the equation for that circle.

5. Determine the equation for the circle that passes through the points $(-5, 3)$, $(-2, 2)$, and $(-10, -2)$.

XIV-2 Show the development of the standard equation of a parabola, either with center (0,0) or center (h,k).

1. Using the figure at the right, show how the equation $x^2 = 4py$ is developed.



2. Show the development of the equation $(y - k)^2 = 4p (x - h)$.

3. Use the definition of eccentricity to explain how the equation $y^2 = 4 px$ is developed.

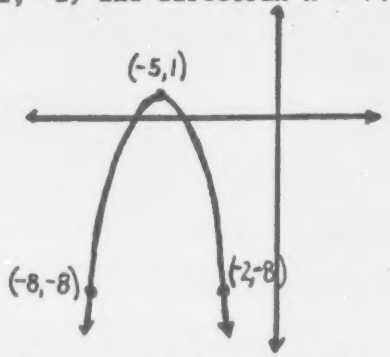
4. Use the definition of eccentricity to show how the equation $(x-h)^2 = 4p (y - k)$ is developed.

XIV-3 **Determine the equation of a parabola from the** information given.

1. **Determine the equation of the parabola with vertex (-2, 1) and focus (-2, 4).**

2. **Determine the equation for a parabola with focus (-1, -2) and directrix x = 7.**

3. **Using the points given in the figure to the right, determine the equation for the parabola.**

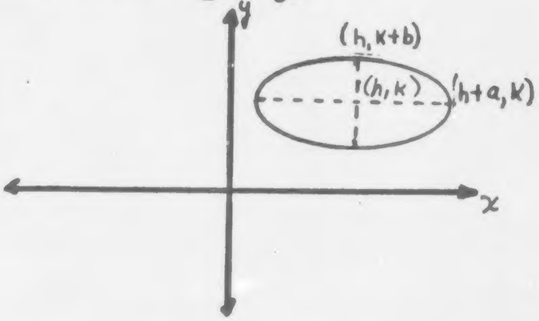


4. **A parabola has its axis of symmetry parallel to the x-axis. If the parabola passes through the points (2, -4), (8, 2), and (26, 8), determine the equation.**

XIV-4 **Show the development of the standard equation of an ellipse, either with center (0,0) or center (h,k).**

1. **Show the development of the equation for an ellipse:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2. **Using the figure at the right, show how** the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is developed.



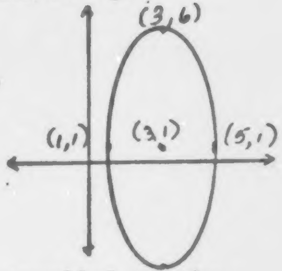
3. **Use the definition of eccentricity to explain how the equation** $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ **is developed.**

4. **Use the definition of eccentricity to show how the equation** $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ **is developed.**

XIV-5 Determine the equation of an ellipse from the information given.

1. Determine the equation for an ellipse with foci ($\pm 3,0$) and $e = \frac{1}{2}$.

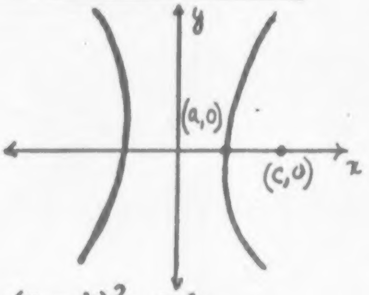
2. Determine the equation of the ellipse shown at the right.



3. Determine the equation of an ellipse with its minor axis parallel to the y-axis and 10 units long, eccentricity 2/3, and center (2,3).

4. If the length of the major axis is 10 units and the foci are (-2, 6) and (-2,-2), determine the equation for the ellipse.

XIV-6 Show the development of the standard equation of a hyperbola, either with center (0,0) or center (h,k).



1. Use the figure at the right to show the development of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

2. Show the development of the equation $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$.

3. Use the definition of eccentricity to explain how the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

is developed.

4. Use the definition of eccentricity to show how the equation $\frac{(y - k)^2}{a^2} - \frac{(x - h^2}{b^2} =$ is developed.

XIV-7 Determine the equation of a hyperbola from the information given.

1. Determine the equation of the hyperbola with vertices (-1,-3) and (-1, 5) which passes through the point (-3,-5).

2. If a hyperbola has foci (-6,-3) and (4,-3) and a vertex at (3,-3), what is its equation?

3. A hyperbola has the transverse axis parallel to the y-axis and 4 units long. If the center is (1,-2) and the eccentricity is 2, determine the equation for the hyperbola.

4. Determine the equation of the hyperbola with center (-3,2), a vertex at (1,2) and an asymptote $y = x + 5$.

XIV-8 **Given the equation of a conic section in the general form**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

a) write the equation in standard form
b) identify the conic, including the degenerate case
c) determine the eccentricity
d) discuss its properties, which may include its center, foci, radius, vertices, axis of symmetry, directrices, asymptotes, major and minor axes, transverse and conjugate axes
e) graph the conic

**For each of the following problems,**

a) **write the equation in standard form**
b) **identify the conic, including the degenerate case**
c) **determine the eccentricity**
d) discuss its properties, which may include its center, foci, radius, vertices, axis of symmetry, directrices, asymptotes, major and minor axes, transverse and conjugate axes
e) **graph the conic**

1. $9x^2 + 4y^2 - 36x + 48y + 144 = 0$

2. $x^2 - y^2 - 6y - 9 = 0$

3. $y^2 + 2x - 8y = -18$

4. $x^2 - y^2 + 6x + 6y + 16 = 0$

XIV-9 **Determine the point(s) of intersection, if any, of two conic sections.**

1. Determine the points of intersection, if any, of

   $x^2 + 2y^2 = 4$ and $x^2 + 3y^2 - 2x = 0$.

2. Determine the points of intersection, if any, of

   $9x^2 + 12y^2 = 108$ and $x^2 = 12(y + 3)$.

3. Determine the points of intersection, if any, of

   $x^2 + y^2 - 4x - 2y + 1 = 0$ and $x^2 + 4y^2 - 4x + 8y = -4$.

4. Determine the points of intersection, if any, of

   $4x^2 - y^2 - 24x + 4y + 48 = 0$ and $x - y^2 + 4y - 8 = 0$.

**XIV-10** Determine the new coordinates of a given point if the origin is moved by translation.

1. After translation of the origin to 0', a point has, in the x'y'-coordinate system, the coordinates (3,-6). What are the xy-coordinates of that point if 0' is (-1,5)?

2. A point P has coordinates (-2,3). The origin is translated to the point 0'(3,-5). What are the x'y'-coordinates of the point P?

3. A point P has coordinates (-5,4). The origin is translated to the point 0'. If the coordinates of P in the x'y'-coordinate system are (-1,7), what are the xy-coordinates of 0'?

4. After translation of the origin to 0', a point has, in the x'y'-coordinate system, the coordinates (8,-6). What are the xy-coordinates of that point if 0' is (-1,-2)?

**XIV-11** Determine the equation of a conic section by translating the origin to the point (h,k). Sketch its graph.

1. Transform the equation $y^2 - 6y + 5 = 4x$ to the x'y' coordinate system by translating the origin to the point (-1,3). Sketch its graph.

2. Transform the equation $x^2 + 2y^2 - 4x + 16y = 2$ to the x'y' coordinate system by translating the origin to the point (2,-4). Sketch its graph.

3. Transform the equation $9x^2 - 4y^2 - 18x + 8y + 4 = 0$ to the x'y' coordinate system by translating the origin to the point (1,1). Sketch its graph.

4. Transform the equation $x^2 + y^2 - 4x + 2y - 20 = 0$ to the x'y' coordinate system by translating the origin to the point (2,1). Sketch its graph.

XIV-12 Given the equation of a conic section in the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, identify the conic, including the degenerate case.

1. Identify the graph of the equation $4x^2 - 5xy + 10y^2 = 16$.
2. Identify the graph of the equation $x^2 - 2xy + y^2 - 8x + 5 = 0$.
3. Identify the graph of the equation $3x^2 - 5xy - 2y^2 + 5x + 4y - 2 = 0$.
4. Identify the graph of the equation $4x^2 + y^2 - 8x + 6y + 13 = 0$.
5. Identify the graph of the equation $3x^2 - 10xy + 3y^2 = 12$.

XIV-13 Determine the angle of rotation necessary to remove the xy-term in the general equation

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Assume that the conic section is not the degenerate case.

1. Determine the angle of rotation necessary to remove the xy-term from the equation
$$5x^2 - 2xy + 5y^2 - 24 = 0.$$
2. Determine the angle of rotation necessary to remove the xy-term from the equation
$$-7x^2 - 12xy + 2y^2 + 10 = 0.$$
3. Determine the angle of rotation necessary to remove the xy-term from the equation
$$2x^2 + \sqrt{3}xy + y^2 = 4.$$
4. Determine the angle of the rotation necessary to remove the xy-term from the equation
$$7x^2 + 4xy + 4y^2 + 6x + 12y - 9 = 0.$$

XIV-14 **Remove the xy-term from an equation of a conic section by a rotation of axes. Sketch the graph of the conic, showing both sets of axes.**

1. Remove the xy-term from the equation $xy = 8$ by a rotation of axes. Sketch the graph, showing both sets of axes.

2. Remove the xy-term from the equation $2x^2 + \sqrt{3}xy + y^2 = 5$ by a rotation of axes. Sketch the graph, showing both sets of axes.

3. Remove the xy-term from the equation $5x^2 - 2xy + 5y^2 - 24 = 0$ by a rotation of axes. Sketch the graph, showing both sets of axes.

XIV-15 Solve verbal problems involving conic sections.

1. A parabolic steel arch is 100 meters high and has its axis vertical and its feet 200 meters apart. How much is the focus above or below the ground?

2. The floor of a building has an elliptical shape, 300 meters long and 200 meters wide. A whisper near one focus can be clearly heard near the other focus. How far apart are the foci?

3. The cross-section of a headlight reflector is a parabola. If the reflector is 12 inches wide and 9 inches deep, how far is it from the vertex to the focus of this parabola.

4. A rectangular lot is surrounded by a walkway that is uniform in width and is 5 feet wide. If the area of the walkway is 780 square feet and the area of the lot is 960 square feet, what are the dimensions of the lot?

2

Pre-Calculus

Unit XIV

ANSWERS

## XIV-1

1. $(x + 2)^2 + (y + 1)^2 = 16$
2. $(x - 2)^2 + (y - \pi)^2 = 3$
3. $(x + 3)^2 + (y - 3)^3 = 9$
4. $(x + 2)^2 + (y + 5)^2 = 80$
5. $(x + 5)^2 + (y + 2)^2 = 25$

## XIV-2

See one of the approved textbooks.

## XIV-3

1. $(x + 2)^2 = 12\ (y - 1)$
2. $(y + 2)^2 = -16\ (x - 3)$
3. $(x + 5)^2 = -(y - 1)$
4. $(y + 4)^2 = 6\ (x - 2)$

## XIV-4

See one of the approved textbooks.

## XIV-5

See one of the approved textbooks.

## XIV-6

1. $\frac{x^2}{36} + \frac{y^2}{27} = 1$
2. $\frac{(x - 3)^2}{4} + \frac{(y - 1)^2}{25} = 1$
3. $\frac{(x - 2)^2}{45} + \frac{(y - 3)^2}{25} = 1$
4. $\frac{(x + 2)^2}{9} + \frac{(y - 2)^2}{25} = 1$

## XIV-7

1. $\frac{(y - 1)^2}{16} - \frac{5(x + 1)^2}{16} = 1$
2. $\frac{(x + 1)^2}{16} - \frac{(y + 3)^2}{9} = 1$
3. $\frac{(y + 2)^2}{4} - \frac{(x - 1)^2}{12} = 1$
4. $(x + 3)^2 - (y - 2)^2 = 16$

## XIV-8

1. a) $\frac{(x - 2)^2}{4} + \frac{(y + 6)^2}{9} = 1$

   b) ellipse

   c) $\frac{\sqrt{5}}{3}$

   d) center $(2,-6)$
   vertices $(4,-6), (0,-6)$
   $(2,-3), (2,-9)$
   foci $(2, -6 \pm \sqrt{5})$
   major axis 6 units
   minor axis 4 units

   e) See graph.

## XIV-8 (continued)

2. a) $x^2 - (y + 3)^2 = 0$

b) two lines $y = x - 3$
$y = x - 3$

c) —
d) —
e) See graph.

3. a) $(y - 4)^2 = -2(x + 1)$

b) parabola

c) 1

d) vertex $(-1,4)$
focus $(-\frac{3}{2},4)$
axis of symmetry $y = 4$
directrix $x = -\frac{1}{2}$

e) See graph.

4. a) $\frac{(y - 3)^2}{16} - \frac{(x + 3)^2}{16} = 1$

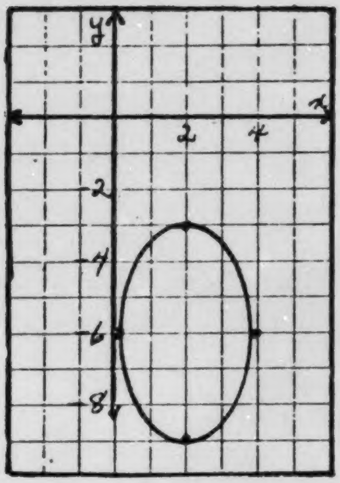
b) hyperbola

c) $\sqrt{2}$

d) center $(-3,3)$
vertices $(-3,-1),(-3,7)$
foci $(-3,3\pm4\sqrt{2})$
asymptotes $y = x + 6$
$y = -x$
transverse axis 8 units
conjugate axis 8 units

e) See graph.

1\.



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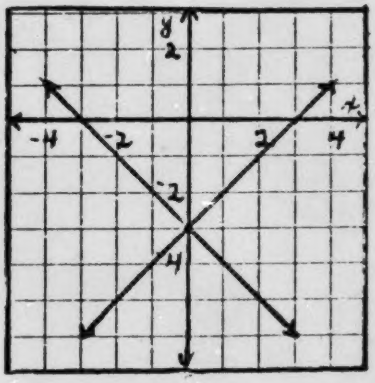

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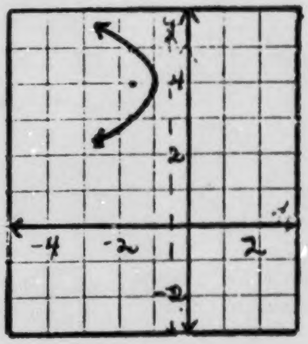

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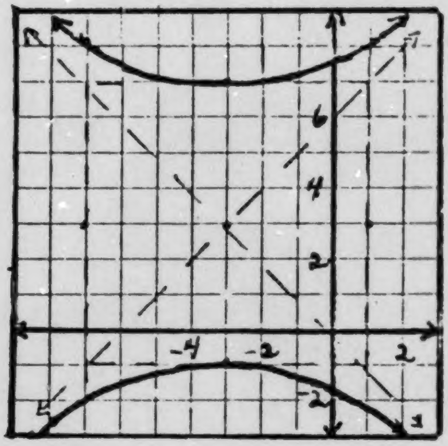


Pre-Calculus

Unit XIV

**ANSWERS**

XIV-9

1. (2,0)
2. (0,-3)
3. $\left(2 \pm \frac{2}{3}\sqrt{5}, -\frac{1}{3}\right)$
4. none

XIV-10

1. (2,-1)
2. (-5,8)
3. (-4,-3)
4. (7,-8)

XIV-11

1. $(y')^2 = 4x'$ where

   $x' = x + 1,\ y' = y - 3$

2. $(x')^2 + 2(y')^2 = 38$ where

   $x' = x - 2,\ y' = y + 4$

3. $9(x')^2 - 4(y')^2 = 1$ where

   $x' = x - 1,\ y' = y - 1$

4. $(x')^2 + (y')^2 = 25$ where

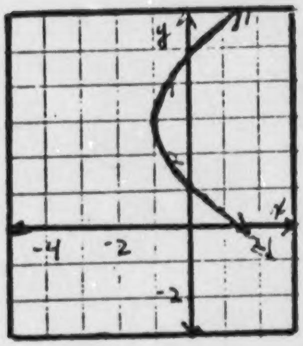
   $x' = x - 2,\ y' = y + 1$

XIV-12

1. ellipse
2. parabola
3. 2 lines, $3x + y = 1$
   $x - 2y = -2$
4. point (1,-3)
5. hyperbola

XIV -11 Graphs

1.



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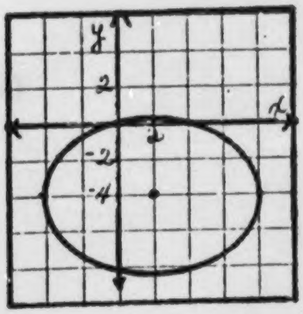

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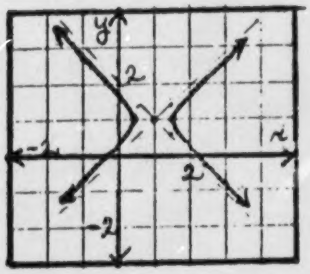

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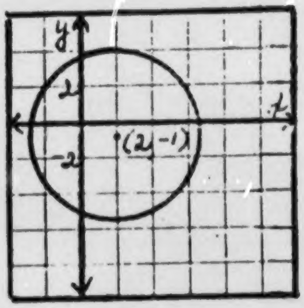


Pre-Calculus

Unit XIV

ANSWERS

### XIV-13

1. $45^\circ = \frac{\pi}{4}$
2. $26^\circ 30'$
3. $\frac{\pi}{6} = 30^\circ$
4. $26^\circ 30'$

### XIV-14

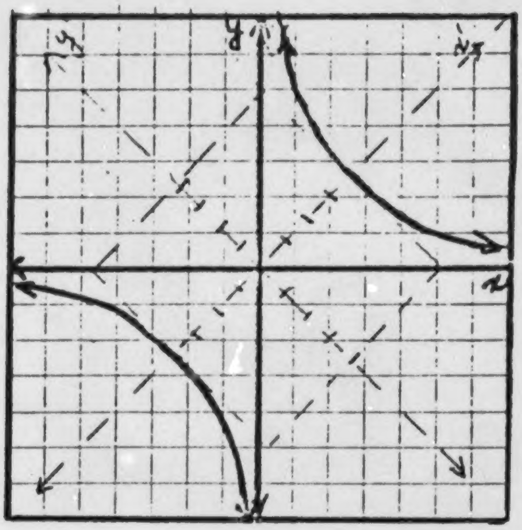
1. $(x')^2 - (y')^2 = 16$
2. $10(x')^2 + 2(y')^2 = 20$
3. $4(x')^2 + 6(y')^2 = 24$

### XIV-15

1. 75 meters above ground
2. 223.6 meters
3. 1 inch
4. 20 feet x 48 feet

### XIV-14 Graphs

1.



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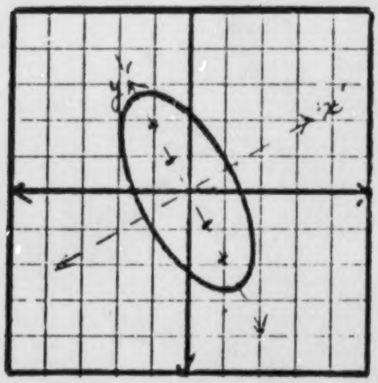

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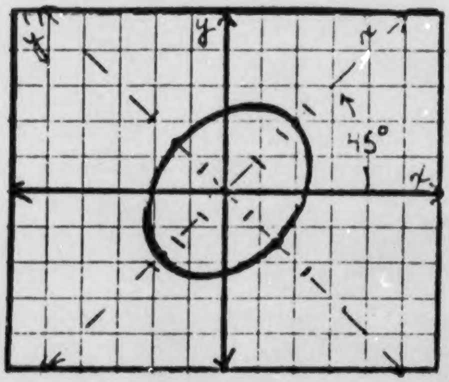

## Unit XV Matrices

### Overview

The primary purpose of this unit is to show the student how a system of two or three linear equations in two or three variables is solved through the use of determinants and matrices. Applications to higher order determinants are accomplished with relatively little difficulty. Matrix algebra is presented and will hopefully benefit those students who will later study linear algebra.

### Suggested Time

10 days

Pre-Calculus

## Unit XV Matrices

### PERFORMANCE OBJECTIVES

1. Given a matrix A,
   - a) determine its dimensions
   - b) identify specific entries of the matrix
   - c) determine its transpose $A^T$
   - d) determine its additive inverse
2. Determine the sum or difference of two or more matrices.
3. Determine the product of a matrix by a scalar.
4. Determine the product of matrices.
5. Evaluate the determinant of a 2 x 2 matrix.
6. Evaluate the determinant of a 3 x 3 (or larger) matrix by one or more of the following methods:
   - a) diagonals
   - b) expansion by a row (column) and cofactors
   - c) reduction by row (column) operations
7. Determine the inverse of a matrix, if it exists.
8. Solve a system of equations using:
   - a) matrices and inverses
   - b) Cramer's Rule
   - c) augmented matrices, Gaussian elimination, or the Gauss-Jordan reduction methods (Part c is optional.)

Unit - XV Matrices

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 482-485<br>487<br>542-543 | | | 408-409 | | |
| 2 | 485-487 | | | 408-411 | | |
| 3 | 483-485 | | | 410-411 | | |
| 4 | 489-492<br>539-542 | | | 411-414 | | 301 |
| 5 | 465<br>502-505 | | | 396-398 | | |
| 6 | | | | 399-401 | | |
| 7 | 449-504 | | | 414-418 | | 302 |
| 8 | 504-505<br>544-555 | | | 402-407<br>418-421 | | 302-303 |
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Pre-Calculus

Unit XV

XV-1 Given a Matrix A,
a) determine its dimensions
b) identify specific entries of the matrix
c) determine its transpose $A^T$
d) determine its additive inverse

1. If matrix $A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & -2 & -8 \end{bmatrix}$,

a) What are the dimensions of A?

b) What number is in the second row, third column?

c) Determine $A^T$.

d) Determine its additive inverse.

2. If matrix $A = \begin{bmatrix} 5 & \sqrt{2} & -2 & -4 \\ -3 & 4 & 10 & 6 \\ 1 & 8 & -1 & 7\pi \end{bmatrix}$,

a) What are its dimensions?

b) Describe the location of the number 6.

c) Determine $A^T$.

d) Determine the additive inverse

3. If matrix $A = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$,

a) What are the dimensions of A?

b) Describe the location of the number 1.

c) Determine $A^T$.

d) Determine the additive inverse.

4. If matrix $A = \begin{bmatrix} 4 & -3 \\ -2 & 9 \end{bmatrix}$,

a) What are the dimensions of A?

b) What number is in the first row, second column?

c) Determine $A^T$.

d) Determine its additive inverse.

Pre-Calculus

Unit XV

XV-2 Determine the sum or difference of two or more matrices.

1. If possible, determine $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 5 & -8 \end{bmatrix} + \begin{bmatrix} -3 & 5 & 0 \\ 9 & -7 & 2 \end{bmatrix}$.

2. If possible, determine $\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ -8 \\ 3 \end{bmatrix}$.

3. If possible, determine $\begin{bmatrix} 1 & 4 & -7 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 6 \\ -2 & 1 & 4 \end{bmatrix}$.

4. If possible, determine $\begin{bmatrix} 3 & -2 \\ 4 & 1 \\ -8 & 9 \end{bmatrix} + \begin{bmatrix} 3 & \sqrt{3} \\ -10 & 6 \\ 1 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ -6 & 4 \\ \sqrt{2} & -2 \end{bmatrix}$.

**XV-3 Determine the product of a matrix by a scalar.**

1. Determine the product: $-2\begin{bmatrix} 6 & 3 \\ 0 & -4 \\ -1 & -6 \end{bmatrix}$

2. Determine the product: $\frac{1}{3}\begin{bmatrix} 5 \\ -\frac{1}{2} \\ -9 \end{bmatrix}$

3. If $A = \begin{bmatrix} 2 & 4 & 9 \\ -2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 1 & 0 \\ 3 & 11 & 5 \end{bmatrix}$, determine 2A - B.

4. If $= \begin{bmatrix} -6 & 4 \\ 6 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -5 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 9 \\ -11 & 8 \end{bmatrix}$, determine A + 3B - 2C.

Pre-Calculus

Unit XV

## XV-4 Determine the product of matrices.

1. Multiply, if possible: $\begin{bmatrix} 3 & -6 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

2. Multiply, if possible: $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & -1 \\ 7 & -2 & 1 \end{bmatrix}$

3. Multiply, if possible: $2\begin{bmatrix} 4 & 6 \\ -2 & 1 \\ 7 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -6 \\ -1 & 8 & 7 \end{bmatrix}$

4. Mutiply, if possible: $\begin{bmatrix} -1 & 3 & 5 \\ 0 & -4 & -2 \\ 1 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 & 0 \\ 1 & 2 & -4 \\ 3 & -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

5. Mutiply, if possible: $\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

XV-5 Evaluate the determinant of a 2 x 2 matrix.

1. Evaluate: $\begin{vmatrix} -3 & -8 \\ 4 & 5 \end{vmatrix}$.

2. Determine the determinant of $\begin{bmatrix} ax & bx \\ a & b \end{bmatrix}$.

3. If $A = \begin{bmatrix} x & 4 \\ 1 & x + 3 \end{bmatrix}$, what is the determinant of A?

4. If $A = \begin{bmatrix} x & x - 4 \\ 2 & 10 \end{bmatrix}$, what is the determinant of A?

XV-6 Evaluate the determinant of a 3 x 3 (or larger) matrix by one or more of the following methods:

a) diagonals
b) expansion by a row (column) and cofactors
c) reduction by row (column) operations

1. Evaluate the determinant, using the diagonal method:

$$\begin{vmatrix} 3 & -1 & 0 \\ -2 & 8 & 4 \\ 1 & 6 & -3 \end{vmatrix}$$

2. Use the cofactors of the second column to evaluate the determinant:

$$\begin{vmatrix} 2 & 4 & -1 \\ -3 & 1 & 5 \\ -4 & -2 & 3 \end{vmatrix}$$

3. Use row (or column) operations to reduce the matrix. Then evaluate the determinant.

$$\begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & -3 \\ 5 & -4 & 9 \end{bmatrix}$$

4. Evaluate the determinant:

$$\begin{vmatrix} -1 & 4 & 3 & -5 \\ 2 & -2 & 6 & -7 \\ -4 & 8 & 1 & -8 \\ -3 & 5 & 7 & -6 \end{vmatrix}$$

XV-7 Determine the inverse of a matrix, if it exists.

1. Determine the inverse, if it exists $\begin{bmatrix} 6 & -3 \\ -4 & -2 \end{bmatrix}$.

2. Determine the inverse, if it exists $\begin{bmatrix} 8 & -12 \\ -2 & 3 \end{bmatrix}$.

3. Determine the inverse, if it exists $\begin{bmatrix} -2 & -1 & -1 \\ 1 & -2 & 3 \\ -1 & -3 & 7 \end{bmatrix}$.

4. Determine the inverse, if it exists $\begin{bmatrix} 1 & 4 & 0 \\ 0 & -2 & 3 \\ 1 & -1 & 5 \end{bmatrix}$.

XV-8 Solve a system of equations by using one of the following methods:

- a) matrices and inverses
- b) Cramer's Rule
- c) augmented matrices, Gaussian elimination, or the Gauss-Jordan reduction methods (Part c is optional.)

1. Use Cramer's Rule to solve the following system of equations:

$$\begin{aligned} x + 2y &= 8 \\ 2x + y + z &= 11 \\ x + y + 2z &= 13 \end{aligned}$$

2. Use matrices and inverses to solve the following system of equations:

$$\begin{aligned} 3x + y &= -1 \\ x + 2y &= 8 \end{aligned}$$

3. Use matrices or determinants to solve the following system:

$$\begin{aligned} 3x + 5y - 2z &= 9 \\ x - 2y + z &= -2 \\ -2x + y - 3z &= 7 \end{aligned}$$

4. Use matrices or determinants to solve the following system:

$$\begin{array}{rrrrcr} 2x & + y & + z & + w & = & 7 \\ 4x & & - 2z & + 2w & = & 4 \\ x & - y & + 2z & - 3w & = & -16 \\ & 3y & - 4z & & = & -5 \end{array}$$

Pre-Calculus

Unit XV

ANSWERS

## XV-1

1. a) 2 x 3

   b) -8

   c) $\begin{bmatrix} 2 & 6 \\ -1 & -2 \\ 5 & -8 \end{bmatrix}$

   d) $\begin{bmatrix} -2 & 1 & -5 \\ -6 & 2 & 8 \end{bmatrix}$

2. a) 3 x 4

   b) second row, fourth column or $A_{2,4}$

   c) $\begin{bmatrix} 5 & -3 & 1 \\ \sqrt{2} & 4 & 8 \\ -2 & 10 & -1 \\ -4 & 6 & 7\pi \end{bmatrix}$

   d) $\begin{bmatrix} -5 & -\sqrt{2} & 2 & 4 \\ 3 & -4 & -10 & -6 \\ -1 & -8 & 1 & -7\pi \end{bmatrix}$

3. a) 3 x 1

   b) third row, first column or $A_{3,1}$

   c) $\begin{bmatrix} 4 & -2 & 1 \end{bmatrix}$

   d) $\begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix}$

4. a) 2 x 2

   b) -3

   c) $\begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$

   d) $\begin{bmatrix} -4 & 3 \\ 2 & -9 \end{bmatrix}$

## XV-2

1. $\begin{bmatrix} -1 & 4 & 4 \\ 9 & -2 & -6 \end{bmatrix}$

2. $\begin{bmatrix} -2 \\ 11 \\ -5 \end{bmatrix}$

3. cannot be added

4. $\begin{bmatrix} 3 & -9+\sqrt{3} \\ 0 & 3 \\ -7-\sqrt{2} & 4 \end{bmatrix}$

## XV-3

1. $\begin{bmatrix} -12 & -6 \\ 0 & 8 \\ 2 & 12 \end{bmatrix}$

2. $\begin{bmatrix} \frac{5}{3} \\ -\frac{1}{6} \\ -3 \end{bmatrix}$

3. $\begin{bmatrix} -2 & 7 & 18 \\ -7 & -9 & -7 \end{bmatrix}$

4. $\begin{bmatrix} -3 & -8 \\ 13 & 2 \end{bmatrix}$

Pre-Calculus

Unit XV

ANSWERS

## XV-4

1. $\begin{bmatrix} -28 \end{bmatrix}$

2. $\begin{bmatrix} -8 & 19 & -5 \\ 30 & -3 & 3 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 120 & 36 \\ -6 & 4 & 38 \\ 20 & -6 & -126 \end{bmatrix}$

4. $\begin{bmatrix} -28 \\ 6 \\ 25 \end{bmatrix}$

5. $\begin{bmatrix} 3x^2 + xy + 4y^2 \end{bmatrix}$

## XV-5

1. 17

2. 0

3. $x^2 + 3x - 4$

4. $8x + 8$

## XV-6

1. $-72 - 4 - 72 + 6 = -142$

2. $-4\begin{vmatrix} -3 & 5 \\ -4 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -3 & 5 \end{vmatrix}$

   $= -44 + 2 + 14 = -28$

3. There are many ways to solve the problem. Here is one solution:

   1) Multiply second column by 2 and add to third column.
   2) Multiply second row by -1 and add to third row.

   The new matrix is $\begin{bmatrix} 4 & -1 & 0 \\ 1 & 2 & 1 \\ 4 & -6 & 0 \end{bmatrix}$.

   The determinant is $-1\begin{vmatrix} 4 & -1 \\ 4 & -6 \end{vmatrix} = 20.$

4. -731

## XV-7

1. $\frac{-1}{24}\begin{bmatrix} -2 & 3 \\ 4 & 6 \end{bmatrix}$

2. no inverse

3. $\frac{1}{25}\begin{bmatrix} -5 & 10 & -5 \\ -10 & -15 & 5 \\ -5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$

4. $\frac{1}{5}\begin{bmatrix} -7 & -20 & 12 \\ 3 & 5 & -3 \\ 2 & 5 & -22 \end{bmatrix}$

Pre-Calculus

Unit XV

ANSWERS

XV-8

1.
$$x = \frac{\begin{vmatrix} 8 & 2 & 0 \\ 11 & 1 & 1 \\ 13 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}} = \frac{-10}{-5} = 2$$

$$y = \frac{\begin{vmatrix} 1 & 8 & 0 \\ 2 & 11 & 1 \\ 1 & 13 & 2 \end{vmatrix}}{-5} = \frac{-15}{-5} = 3$$

$$z = 4$$

2.
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -8 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

3. $x = 1$, $y = 0$, $z = -3$

4. $x = -1$, $y = 1$, $z = 2$, $w = 6$

Pre-Calculus - Analytic Geometry

## Unit XVI Polar Coordinates

### Overview

The student is familiar with polar coordinates from the previous work with DeMoivre's Theorem. However, the rest of the material is new to the student. The unit begins with plotting points and then goes on to graphing equations (by substitution of θ values and then by considering aides such as symmetry, intercepts, etc.) and to solving systems of polar equations.

### Suggestions to the Teacher

The instruction and development of polar coordinates is similar to the development of rectangular coordinates. It might be helpful to ask the students to recall the development of rectangular coordinates as well as the graphing and solving of rectangular equations. When solving the system of polar equations, the student should be aware of the difference between analytical solutions and geometric solutions. Analytic Geometry, 5th Edition, by Gordon Fuller has a very comprehensive treatment of polar coordinates. Algebra, Trigonometry, and Analytic Geometry, by Rees and Sparks has a good section on transforming from rectangular to polar coordinates and vice versa. Objective 8 may be optional depending on teacher preference and textbook used.

### Suggested Time

7 days

310

Pre-Calculus

## Unit XVI Polar Coordinates

### PERFORMANCE OBJECTIVES

1. Plot points when given the polar coordinates of the point $(r,\theta)$.
2. Represent a given point $(r,\theta)$ in at least three equivalent forms where $-2\pi \leq \theta \leq 2\pi$.
3. Using several aids in graphing such as symmetry, intercepts, extent of r, etc., sketch the graph given the equation written in polar form.
4. Determine the rectangular equation when given the polar equation.
5. Determine the polar equation when given the rectangular equation.
6. Sketch the graphs of conic sections when given the equation in polar form.
7. Sketch the graphs of two equations written in polar form, and determine the coordinates of the point(s) of intersection.
8. Determine the equation of the conic when given specific information. (optional)

Unit XVI - Polar Coordinates

## CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 196-199 | 363-366 | 158-162 | 482-484 |  | 243-244 |
| 2 | 196-199 | 363-366 | 158-162 | 482-484 |  | 243-244 |
| 3 | 200-204 | 366-387 | 166-178 | 484-489 |  | 251-256 |
| 4 | 198-199 | 366-370 | 163-166 | 490-492 |  | 249-250 |
| 5 | 198-199 | 366-370 | 163-166 | 490-492 |  | 249-250 |
| 6 | 204 | 370 | 182-187 | 494-496 |  | 259-262 |
| 7 | 204 | 381-383 | 190-195 | 493-494 |  | 257-258 |
| 8 |  | 387-390 |  | 494-495 |  | 262 |
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**XVI-1** Plot points when given the polar coordinates of the point $(r,\theta)$.

1. Plot the following points:

   a) (3,30°)
   b) (4,120°)
   c) (-6,20°)
   d) (-5,-140°)
   e) (2,270°)

2. Plot the following points:

   a) (5,150°)
   b) $(-3,\frac{\pi}{2})$
   c) (4,-120°)
   d) (-2,-200°)
   e) (3,-300°)

**XVI-2** Represent a given point $(r, \theta)$ in at least three equivalent forms where $-2\pi \le \theta \le 2\pi$.

Represent each of the following points in three equivalent forms where $-2\pi \le \theta \le 2\pi$.

1. (3,20°) = ______________ = ______________ = ______________

2. (-4,155°) = ______________ = ______________ = ______________

3. (5,200°) = ______________ = ______________ = ______________

4. (2,-300°) = ______________ = ______________ = ______________

5. (-6,-75°) = ______________ = ______________ = ______________

6. $(7,-\frac{7\pi}{6})$ = ______________ = ______________ = ______________

XVI-3 Using several aids in graphing such as symmetry, intercepts, extent of r, etc., sketch the graph given the equation written in polar form.

Sketch each of the following curves:

1. $r = 2 - 4 \sin \theta$
2. $r = -6 \cos \theta$
3. $r = 5 \sin 3\theta$
4. $r^2 = 25 \cos 2\theta$
5. $r = 3 \csc \theta$
6. $r = -4 + 2 \cos \theta$

XVI-4 Determine the rectangular equation when given the polar equation.

Transform each equation from polar form into its rectangular form.

1. $r = 10 \cos \theta$
2. $r^2 \cos 2\theta = 25$
3. $r = 4 \tan \theta \cdot \sec \theta$
4. $r = 2\sqrt{2} \sin (\theta + 45°)$
5. $r = \dfrac{8}{5 + 3 \cos \theta}$
6. $r = \tan^2 \theta \cdot \sec \theta$
7. $r = 2 - 3 \cos \theta$
8. $r = 4 \sin^2 \frac{1}{2}\theta$

XVI-7 Sketch the graphs of two equations written in polar form, and determine the coordinates of the point(s) of intersection.

Sketch the graphs of the following equations and determine the point(s) of intersection.

1. $\begin{cases} r = 2 \sec \theta \\ r = 2\sqrt{3} \csc \theta \end{cases}$

2. $\begin{cases} r = 4 \sin \theta \\ r = 3 \csc \theta \end{cases}$

3. $\begin{cases} r = \sin \theta \\ r = \sin 2\theta \end{cases}$

4. $\begin{cases} r = \dfrac{-2}{1 + \cos \theta} \\ r = \dfrac{6}{1 + \cos \theta} \end{cases}$

5. $\begin{cases} r = \dfrac{3}{2 - \cos \theta} \\ r = 4 \cos \theta \end{cases}$

6. $\begin{cases} r = \sin 3\theta \\ r = \frac{1}{2}\sqrt{2} \end{cases}$

XVI-5 Determine the polar equation when given the rectangular equation.

Transform the following equations into polar coordinates:

1. $2x - 7y - 5 = 0$
2. $x^2 = 4py$
3. $x^2 + y^2 = a^2 x$
4. $(x^2 + y^2)(x - a)^2 = x^2 b^2$
5. $y^2 = \frac{x^3}{a - x}$
6. $(x^2 + y^2 - ax)^2 = b^2 (x^2 + y^2)$
7. $y^2 = \frac{x^2 (3a - x)}{a + x}$
8. $\frac{(x - 5)^2}{16} - \frac{y^2}{9} = 1$

XVI-6 Sketch the graphs of conic sections when given the equation in polar form.

Sketch each of the following conics:

1. $r = \frac{4}{1 - \cos\theta}$
2. $r = \frac{4}{1 + 2\cos\theta}$
3. $r = \frac{10}{4 + 2\sin\theta}$
4. $r = \frac{8}{4 - \sin\theta}$

**XVI-8** Determine the equation of the conic when given specific information (optional).

Find the equation of the conics satisfying the following conditions. In all cases, a focus is at the pole.

1. Parabola with directrix $r = -2 \sec \theta$
2. Ellipse, eccentricity $\frac{3}{5}$, other focus at (6,0°)
3. Hyperbola, eccentricity $\frac{5}{4}$, center at (5,0°)
4. Parabola with vertex at (2,270°)
5. Ellipse with ends of minor axis at (8,60°) and (8,120°)

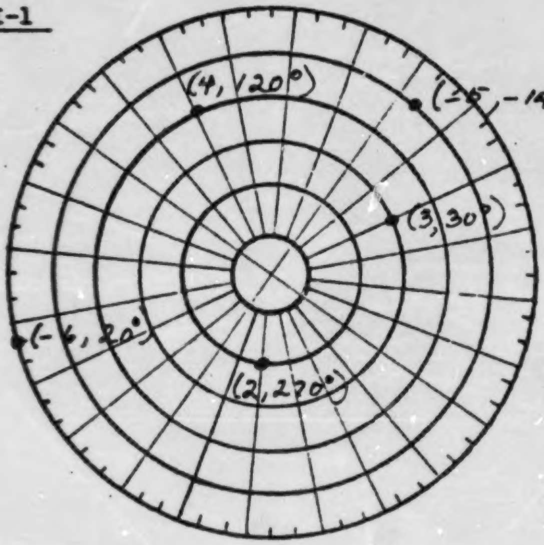
Pre-Calculus

Unit XVI

## ANSWERS

### XVI-1

1.



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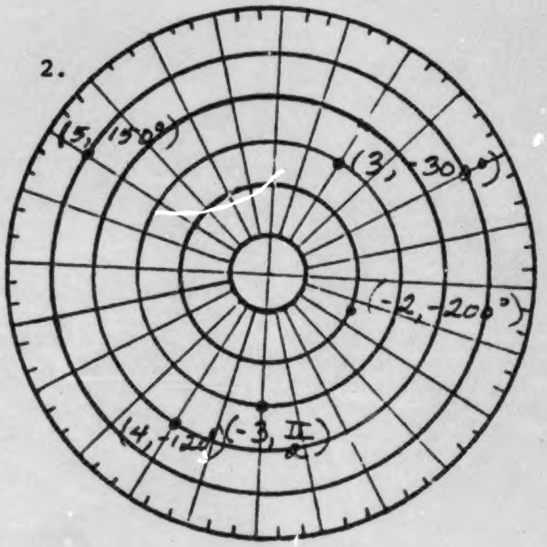


### XVI-2

1. $(3, 20°) = (3, -340°) = (-3, 200°) = (-3, -160°)$

2. $(-4, 155°) = (4, -25°) = (4, 335°) = (-4, -205°)$

3. $(5, 200°) = (5, -160°) = (-5, 20°) = (-5, -340°)$

4. $(2, -300°) = (2, 60°) = (-2, -120°) = (-2, 240°)$

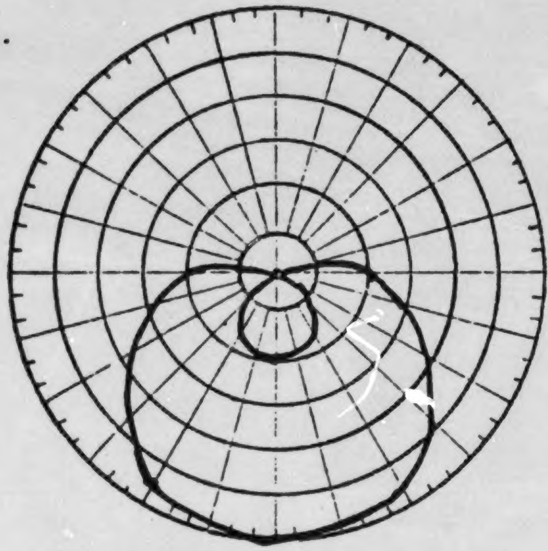
5. $(-6, -75°) = (6, 105°) = (6, -255°) = (-6, 285°)$

6. $(7, \frac{-7\pi}{6}) = (7, \frac{5\pi}{6}) = (-7, -\frac{\pi}{6}) = (-7, \frac{11\pi}{6})$

ANSWERS

XVI-3

1.



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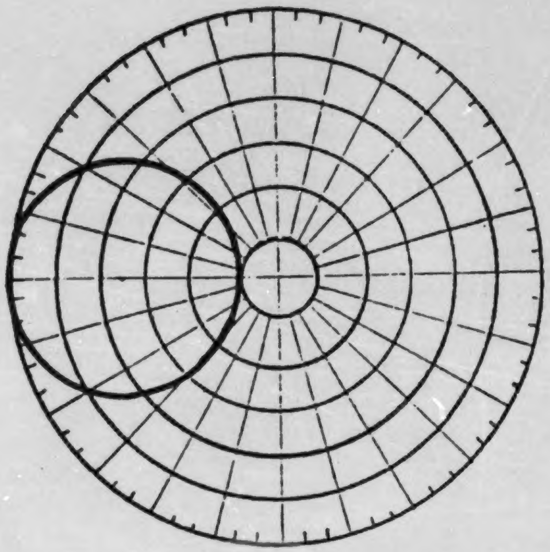

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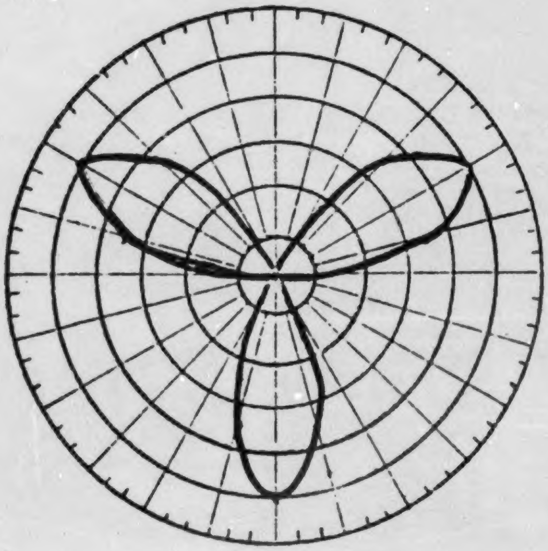

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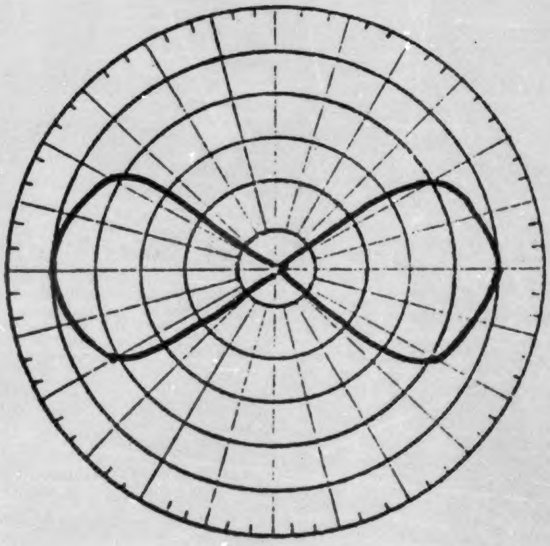


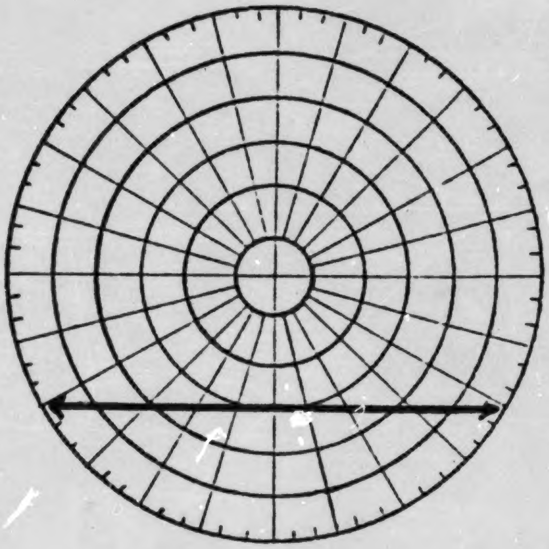
**Pre-Calculus**

**UNIT XVI**

**ANSWERS**

**XVI-3** (continued)

5.



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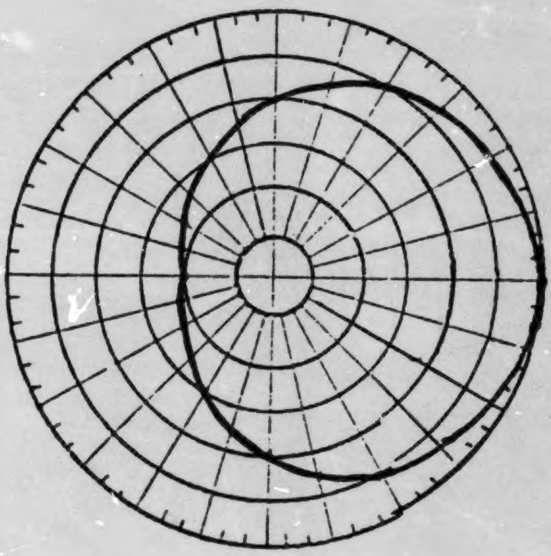


**XVI-4**

1. $x^2 + y^2 - 10x = 0$
2. $x^2 - y^2 = 25$
3. $x^2 = 4y$
4. $x^2 + y^2 - 2x - 2y = 0$
5. $16x^2 + 25y^2 + 48x - 64 = 0$
6. $y^2 = x^3$
7. $(x^2 + y^2 + 3x)^2 = 4(x^2 + y^2)$
8. $(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$

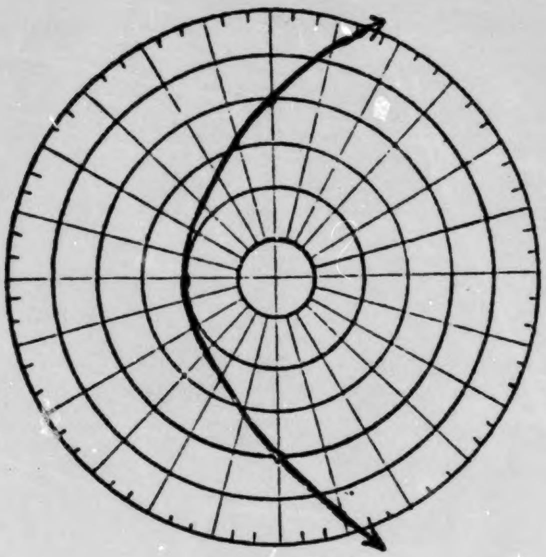
**XVI-5**

1. $r = \dfrac{5}{2\cos\theta - 7\sin\theta}$
2. $r = 4p\sec\theta\tan\theta$
3. $r = a^2\cos\theta$
4. $r = a\sec\theta \pm b$
5. $r = a\sin\theta\tan\theta$
6. $r = a\cos\theta \pm b$
7. $r = a(4\cos\theta - \sec\theta)$
8. $r = \dfrac{9}{5\cos\theta \pm 4}$

**ANSWERS**

**XVI-6**

1.



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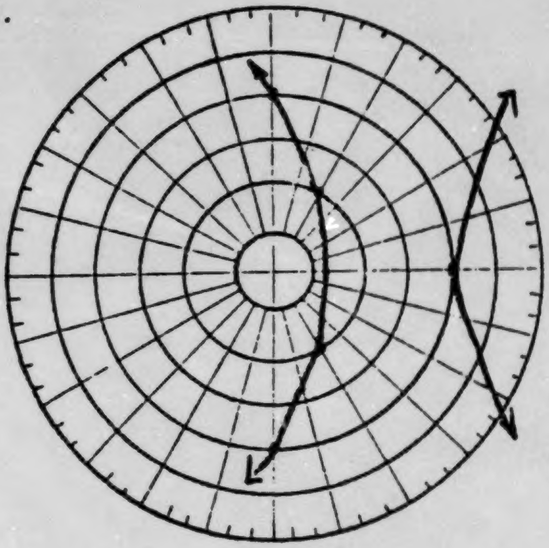

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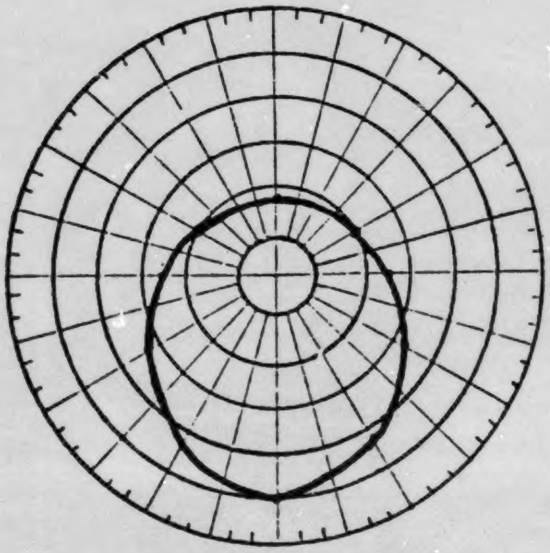

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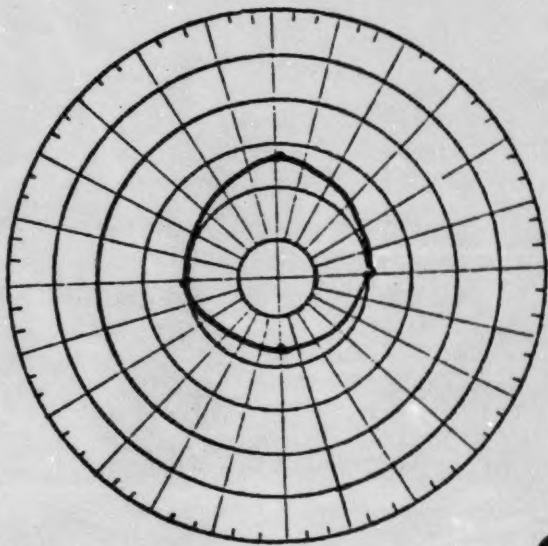
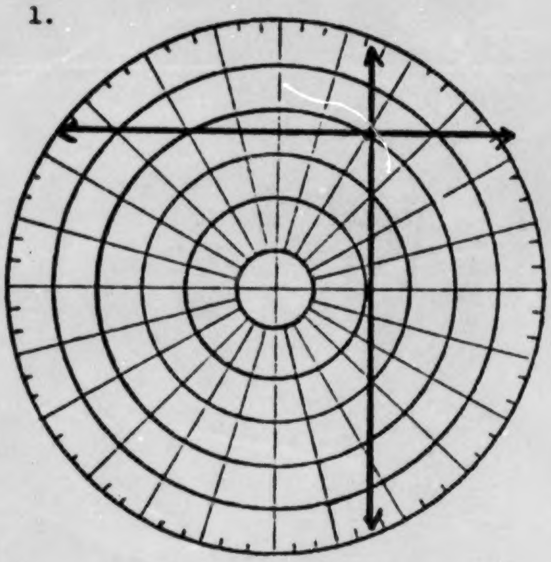


## ANSWERS

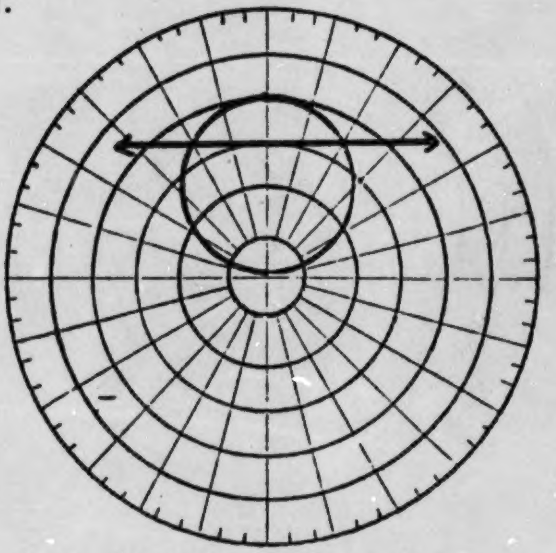
### XVI-7

1.



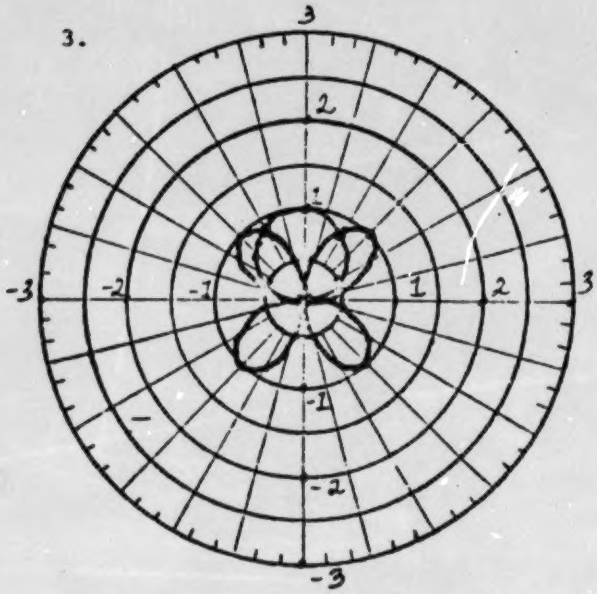
(4, 60°)

2.



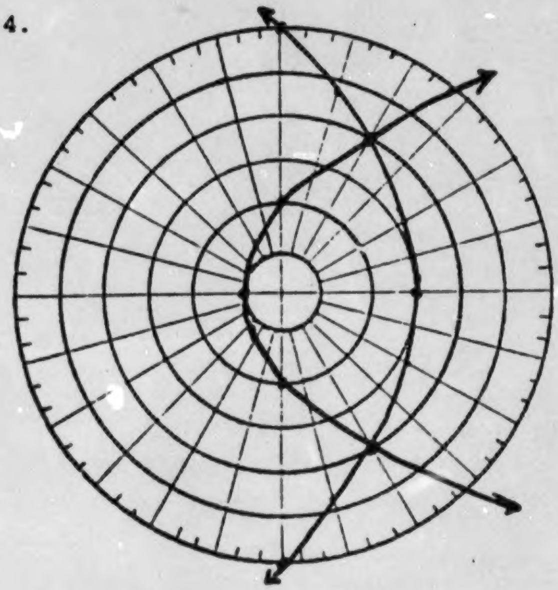
(2 $\sqrt{3}$, 60°), (2 $\sqrt{3}$, 120°)

3.



$(\frac{\sqrt{3}}{2}, 60°)$, $(\frac{\sqrt{3}}{2}, 120°)$, pole

4.

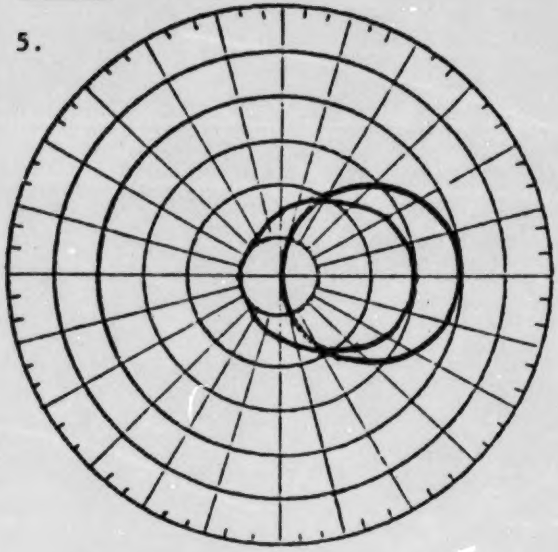


(4, 60°),(4, -60°)

## ANSWERS

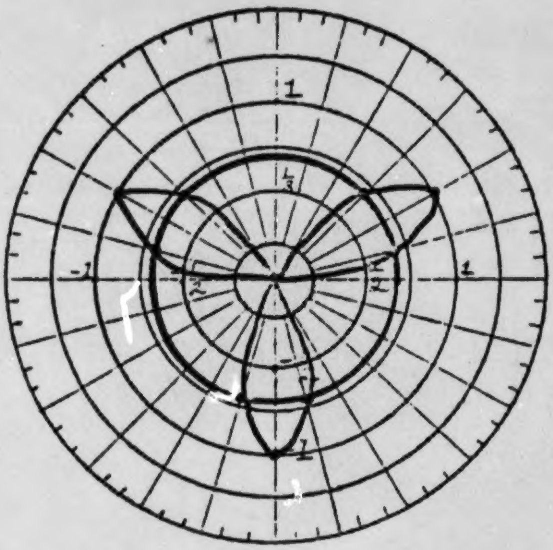
### XVI-7

5.



(2, 60°),(2,- 60°)

6.



$(\frac{1}{2}\sqrt{2}, 15°)$, $(\frac{1}{2}\sqrt{2}, 45°)$, $(\frac{1}{2}\sqrt{2}, 135°)$
$(\frac{1}{2}\sqrt{2}, 165°)$, $(\frac{1}{2}\sqrt{2}, 255°)$, $(\frac{1}{2}\sqrt{2}, 285°)$

### XVI-8

1. $r = \frac{2}{1 - \cos\theta}$

2. $r = \frac{16}{5 - 3\cos\theta}$

3. $r = \frac{9}{4 + 5\cos\theta}$

4. $r = \frac{4}{1 - \sin\theta}$

5. $r = \frac{4}{2 - \sqrt{3}\sin\theta}$

Pre-Calculus - Analytic Geometry

## Unit XVII Parametric Representation of Curves

### Overview

Parametric equations for lines were studied earlier in the semester. This section continues with this study and relates the use of parametric equations to curves in the plane and in space. Practical applications are also used to enhance the subject.

### Suggestions to the Teacher

The objectives in this unit may be incorporated in Unit X or in any other unit as appropriate as an alternative to presentation as a separate unit.

Depending on the students in the class, the teacher may wish to eliminate the parameter from a set of equations (Objective 2) before graphing them. The idea in Objective 3 should be emphasized. Problems on the path of a projectile are included in this unit; other applications of parametric equations, such as cycloids and involutes of circles, are left to the teacher's discretion. Objective 6 is optional, with the answers to these curves found in most textbooks that cover the topic on parameters. Since most of the textbooks do not present drawing a curve in space using parametric equations, the last objective is also considered optional.

### Suggested Time

5 days

## Unix XVII Parametric Representation of Curves

PERFORMANCE OBJECTIVES

1. Sketch the graph represented by a set of parametric equations.
2. Eliminate the parameter from a set of parametric equations to obtain an equivalent rectangular equation.
3. Determine whether eliminating a parameter produces a graph that is equivalent to the graph for the parametric equations. If it is not equivalent, describe the portion of the graph covered by the parametric equations.
4. Determine the parametric representations for a rectangular equation of a curve.
5. Solve verbal problems involving the path of a projectile.
6. Sketch the graphs of special curves (cycloids, Witch of Agnesi, folium of Descartes, cissoid of Diocles, etc.). (optional)
7. Sketch the graph of a curve in space, given its parametric equations. (optional)

Unit - XVII - Parametric Representation of Curves

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | | | 200-203 | 508-511 | | 231-235 |
| 2 | | | 196-203 | 512-513 | | 232-235 |
| 3 | | | 200-204 | 512-513 | | 232-235 |
| 4 | | | 204 | 514-517 | | 233-235 |
| 5 | | | 204-206 | 518-520 | | |
| 6 | | | 206-209 | 518-520 | | |
| 7 | 647-650 | | | 324-326 | | |
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**XVII-1 Sketch the graph represented by a set of parametric equations.**

1. Sketch the graph of $x = \cos t,\ y = 2 \sin t$.

2. Sketch the graph of $x = 2 + t,\ y = -6 + 5t$.

3. Sketch the graph of $x = t^2 + 1,\ y = t,\ t \geq 0$.

4. Sketch the graph of $x = \frac{1}{t + 2},\ y = t + 1,\ t \leq 0$.

5. Sketch the graph of $x = 2 \cos t + 3 \sin t$,
$y = 2 \sin t - 3 \cos t$.

**XVII-2** Eliminate the parameter from a set of parametric equations to obtain an equivalent rectangular equation.

1. Eliminate the parameter from the equations

   $x = 3 + 2t$ and $y = 2 - t$.

2. Eliminate the parameter from the equations

   $x = 2 \sin t$ and $y = 5 \cos t$.

3. Eliminate the parameter from the equations

   $x = \sin^2 \theta - 1$ and $y = \cos \theta - 1$.

4. Eliminate the parameter from the equations

   $x = \cos 2\theta$ and $y = \cos \theta$.

5. Eliminate the parameter from the equations

   $x = t^2 - 2t - 1$ and $y = t^2 - t + 2$.

XVII-3 Determine whether eliminating a parameter produces a graph that is equivalent to the graph for the parametric equations. If it is not equivalent, describe the portion of the graph covered by the parametric equations.

1. Determine whether eliminating the parameter from $x = \cos t$ and $y = \sin^2 t$ produces a graph that is equivalent to the graph of the parametric equations. If not, describe the portion of the graph given by the parametric equations.

2. Determine whether eliminating the parameter from $x = t^2$ and $y = t$ produces a graph that is equivalent to the graph of the parametric equations. If not, describe the portion of the graph given by the parametric equations.

3. Determine whether eliminating the parameter from $x = 3 - 2t^2$ and $y = t^2 + 4$ produces a graph that is equivalent to the graph of the parametric equations. If not, describe the portion of the graph given by the parametric equations.

4. Determine whether eliminating the parameter from $x = 2 + \cos \theta$ and $y = -2 - \cos \theta$ produces a graph that is equivalent to the graph of the parametric equations. If not, describe the portion of the graph given by the parametric equations.

**XVII-4** Determine the parametric representations for a rectangular equation of a curve.

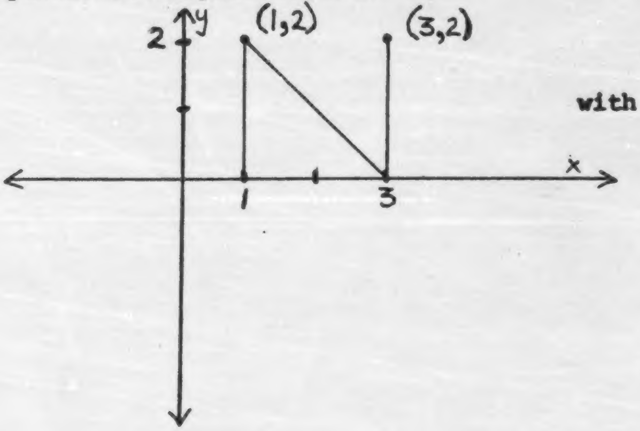
1. Determine the parametric representations for the equation $(x + 1)^2 + 16(y - 2)^2 = 16$. Use the substitution $x = -1 + 4 \sin \theta$.

2. Determine the parametric representations for the equation $5x - 3y = 13$. Use the substitution $y = 5t - 1$.

3. Determine the parametric representations for the equation $4x^2 + 9y^2 = 36$.

4. Determine the parametric representations for the equation $x^2 y = y - x$.

5. Determine the parametric representations for



with $t \in [0,2]$.

**XVII-5 Solve verbal problems involving the path of a projectile.**

According to a law in physics, a projectile fired at an angle $\theta$ moves along a path given by the equations $x = v_0 t \cos \theta$ and $y = v_0 t \sin \theta - 16 t^2$, where $v_0$ = initial velocity in feet per second, $t$ is time in seconds, and the only force acting upon the projectile is gravity.

1. If $v_0 = 200$ feet per second and $\theta = 30^\circ$, what are the coordinates $(x,y)$ of the projectile for $t = 2$?

2. If $v_0 = 200$ feet per second and $\cos \theta = \frac{4}{5}$, at what time $t$ will the projectile hit the ground?

3. If $v_0 = 200$ feet per second and $\cos \theta = \frac{4}{5}$, where does the projectile hit the ground?

4. What is the maximum height of the projectile given in part c?

**XVII-6** Sketch the graphs of special curves (cycloids, Witch of Agnesi, folium of Descartes, cissoid of Diocles, etc.). (optional)

1. Sketch the folium of Descartes, given by the parametric equations

$$\left.\begin{aligned} x &= \frac{3t^2}{1+t^3} \\ y &= \frac{3t}{1+t^3} \end{aligned}\right\} \quad t \neq -1.$$

2. Sketch the Witch of Agnesi, given by the parametric equations

$$x = 4\cot\theta$$
$$y = 4\sin^2\theta.$$

3. Sketch the curtate cycloid, given by the parametric equations

$$x = 9\theta - 4\sin\theta$$
$$y = 9 - 4\cos\theta.$$

4. Sketch the cissoid of Diocles, given by the parametric equations

$$x = 2\sin^2\theta$$
$$y = 2\sin^2\theta \cdot \tan\theta.$$

XVII-7 Sketch the graph of a curve in space, given its parametric equations. (optional)

1. Sketch the graph of the curve given by

$$\begin{aligned} x &= 2 \sin \phi \cos \theta \\ y &= 2 \sin \phi \sin \theta \\ z &= 2 \cos \phi. \end{aligned}$$

2. Sketch the graph of the curve given by

$$\begin{aligned} x &= 4 - 2s - t \\ y &= s \\ z &= t. \end{aligned}$$

3. Sketch the graph of the curve given by

$$\begin{aligned} x &= s \\ y &= \sqrt{8 - 2s^2 - 4t^2} \\ z &= t, \ t \geq 0. \end{aligned}$$

4. Sketch the graph of the curve given by

$$\left.\begin{aligned} x &= \sqrt{v} \ \cos \theta \\ y &= \sqrt{v} \ \sin \theta \\ z &= v, \ v \geq 0 \end{aligned}\right\} \quad 0 \leq \theta \leq 2\pi.$$

**Pre-Calculus**

**Unit XVII**

## ANSWERS

### XVII-1

See graphs.

### XVII-2

1. $x + 2y = 7$
2. $\frac{x^2}{4} + \frac{y^2}{25} = 1$
3. $(y + 1)^2 = -x$
4. $\frac{1}{2}(x + 1) = y^2$
5. $x^2 - 2xy + y^2 + 7x - 8y + 14 = 0$

### XVII-5

1. $(200\sqrt{3}, 136)$
2. 7½ seconds
3. 1200 feet away
4. 225 feet

### XVII-6

See graphs given in textbooks.

### XVII-7

See graphs.

### XVII-3

1. the portion of the parabola $x^2 + y = 1$ where $y \geq 0$
2. the graphs are equivalent.
3. the portion of the line $x + 2y = 11$, beginning at the point (3,4), extending upward to the left
4. a line segment of $x + y = 0$, where the end points of the segment are (3,-3) and (1,-1)

### XVII-4

Other answers may be acceptable.

1. $x = -1 + 4\sin\theta$
   $y = 2 + \cos\theta$
2. $x = 2 + 3t$
   $y = -1 + 5t$
3. $x = 3\cos\theta$
   $y = 2\sin\theta$
4. $x = t$
   $y = \frac{-t}{t^2 - 1}$
5. $\left.0 \leq t \leq \frac{2}{3}\right\}$ $x = 1$, $y = 3t$

   $\left.\frac{2}{3} \leq t \leq \frac{4}{3}\right\}$ $x = 3t - 1$, $y = -3t + 4$

   $\left.\frac{4}{3} \leq t \leq 2\right\}$ $x = 3$, $y = 3t - 4$

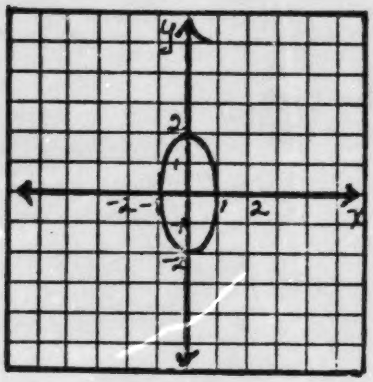
Pre-Calculus

Unit XVII

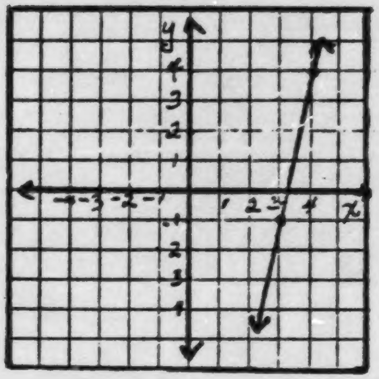
ANSWERS (continued)

XVII-I

1.



2.



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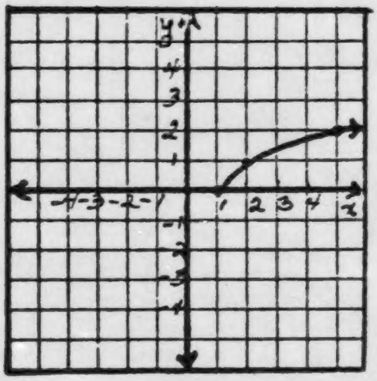

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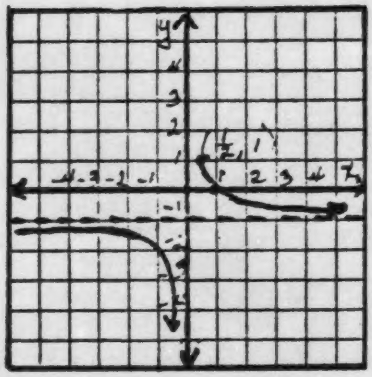
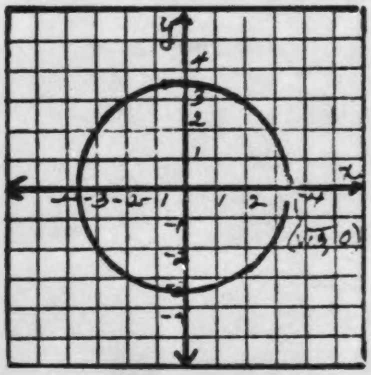
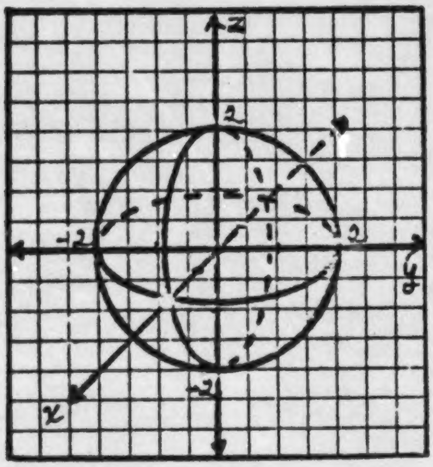


5.



XVII-7

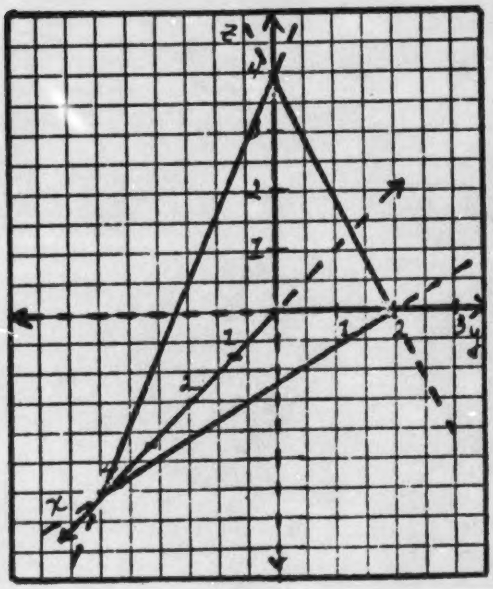
1.



**Sphere**

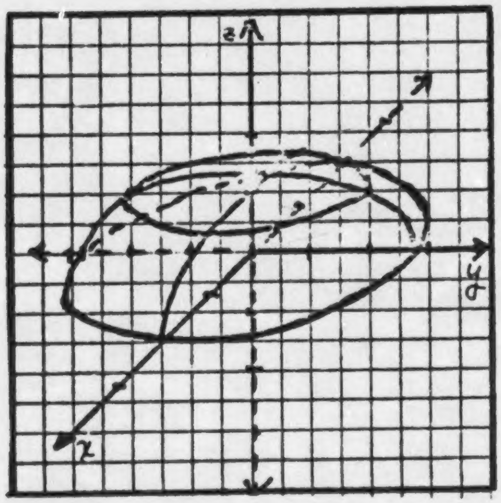
Center: (0, 0)
Radius: 2

2.



**Plane**

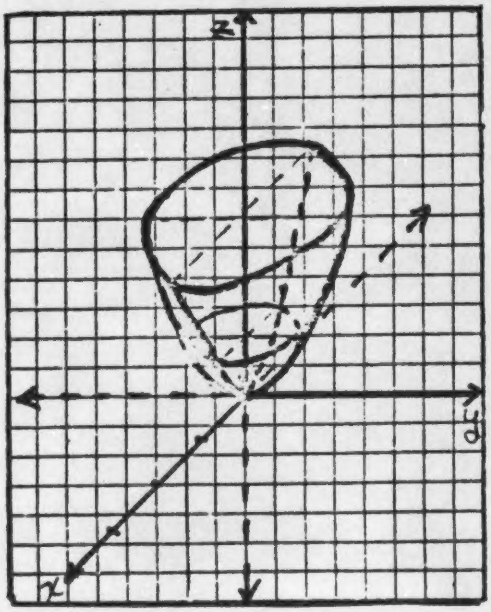
3.



**Ellipsoid**

(Upper half)

4.



**Paraboloid**

Pre-Calculus - Analytic Geometry

## Unit XVIII Surfaces

### Overview

The topics studied in the unit are extensions of some ideas already presented in Pre-Calculus. Equations and graphs of spheres, quadric surfaces, and surfaces of revolution are outgrowths of the study on circles and conic sections. Two new coordinated systems, spherical and cylindrical, are presented at this time.

### Suggestions to the Teacher

Objectives 1, 2, and 3 contain basic problems on spheres and cylinders. These objectives, as well as those concerned with spherical and cylindrical coordinates, could be discussed with some degree of ease and speed. Other areas of discussion and graphing methods may cause some difficulty for students. More time on quadric surfaces and surfaces of revolution will probably be needed. Any or all of the items given in Objectives 4 and 5 may be covered and/or tested. The presentation of the material in this unit is fairly good in the Analytic Geometry by Fuller and Analytic Geometry by Protter, with an excellent discussion given in Modern Analytic Geometry by Wooton.

Objectives 6-9 are optional.

### Suggested Time

8 days

## Unit XVIII Surfaces

### PERFORMANCE OBJECTIVES

1. Determine the equation of a sphere, using the information given.

2. Given an equation of a sphere, determine its center and radius. Sketch its graph.

3. Sketch the portion of the cylinder described.

4. Given the equation of a quadric surface,

    a) identify the surface
    b) discuss any symmetry
    c) describe the traces on the coordinate planes
    d) determine the intercepts on the coordinate axes
    e) sketch the graph

5. Given a surface of revolution,

    a) determine the traces on the coordinate planes
    b) identify the axis (axes) of revolution
    c) sketch the graph

Optional:

6. Determine the cylindrical coordinates of a point with rectangular coordinates, and vice versa.

7. Given a rectangular equation, determine the equation in cylindrical coordinates, and vice versa.

8. Determine the spherical coordinates of a point with rectangular coordinates, and vice versa.

9. Given a rectangular equation, determine the equation in spherical coordinates and vice versa.

## Unit XVIII - Surfaces

CROSS-REFERENCES

| Objectives | Coxford | Crosswhite | Fuller | Shanks | Sorgenfrey | Wooton |
|---|---|---|---|---|---|---|
| 1 | 628, 631 | 266-268 | - | 433-435 |  | 345-350 |
| 2 | 628, 631 | 266-268 | 251 | 433-435 |  | 345-350 |
| 3 | 629-631 | 291 | - | 430-433 |  | 350-352<br>355 |
| 4 | 635-642 | 290-291 | 239-252 | 467-470 |  | 355-365 |
| 5 | 632-635 |  | 236-239 |  |  | 352-355 |
| 6 | 642-647 |  |  | 496-498 |  | 370-374 |
| 7 | 642-647 |  |  | 496-498 |  | 370-374 |
| 8 | 642-647 |  |  | 498-500 |  | 370-374 |
| 9 | 642-647 |  |  | 498-500 |  | 370-374 |
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Pre-Calculus

Unit XVIII

**XVIII-1 Determine the equation of a sphere, using the information given.**

1. The center of a sphere is (3, 1, -2). If the radius is 4, determine the equation of the sphere.

2. Determine the equation of a sphere if its center is (4, -6, 0) and the radius is $\sqrt{3}$.

3. Determine the equation of a sphere if its center is (-2, 3, -5) and the sphere contains the point (-5, -4, -1).

4. Determine the equation of a sphere if it passes through the points (2, 1, -1) (5, -6, 1), (3, 3, 2), and (-4, -7, 11).

XVIII-2 Given an equation of a sphere, determine its center and radius. Sketch its graph.

1. Determine the center and radius of the sphere
$$(x + 1)^2 + (y - 1)^2 + (z + 2)^2 = 4.$$
Sketch its graph.

2. Determine the center and radius of the sphere $x^2 + y^2 + z^2 = 12$. Sketch its graph.

3. Determine the center and radius of the sphere
$$x^2 + y^2 + z^2 - 2x - 2z = 34.$$
Sketch its graph.

4. Determine the center and radius of the sphere
$$2x^2 + 2y^2 + 2z^2 - 8y + 12z + \frac{3}{2} = 0.$$
Sketch its graph.

## XVIII-3 Sketch the portion of the cylinder described.

1. Sketch the cylinder: $x^2 = 8(y + 1)$ and $-2 \leq z \leq 2$.

2. Sketch the cylinder: $x^2 + y^2 - 4x + 2y + 1 = 0$ and $0 \leq z \leq 3$.

3. Sketch the cylinder: $y^2 + 4z^2 = 16$ and $-1 \leq x \leq 2$.

4. Sketch the cylinder: $x^2 + z^2 - 4x - 4z + 4 = 0$ if a generator of the cylinder is the line $y = x - 2$ and $0 \leq y \leq 4$.

| | |
|---|---|
| XVIII-4 Given the equation of a quadric surface, | a) identify the surface<br>b) discuss any symmetry<br>c) describe the traces on the coordinate planes.<br>d) determine the intercepts on the coordinate axes<br>e) sketch the graph |

For each of the following equations,

a) identify the quadric surface
b) discuss any symmetry
c) describe the traces on the coordinate planes
d) determine the intercepts on the coordinate axes
e) sketch the graph

1. $4z = 2x^2 + y^2$
2. $9x^2 - 8y^2 + 4z^2 = 72$
3. $64x^2 + y^2 + 16z^2 = 64$
4. $x^2 + y^2 + z^2 - 2x = 0$

**XVIII-5** Given a surface of revolution,
a) determine the traces on the coordinate planes
b) identify the axis (axes) of revolution
c) sketch the graph

For each surface of revolution,

- a) determine the traces on the coordinate planes
- b) identify the axis (axes) of revolution
- c) sketch the graph

1. $4x^2 + 4y^2 - z^2 = 16$
2. $2x^2 + y^2 + 2z^2 - 4y = 0$
3. $4(x^2 + y^2) + z^2 = 16$
4. $x^2 + 4y^2 - z^2 = 0$

**XVIII-6** Determine the cylindrical coordinates of a point with rectangular coordinates, and vice versa.

1. Determine the cylindrical coordinates of the point with rectangular coordinates $(-8, 0, 6)$.

2. Determine the cylindrical coordinates of the point with rectangular coordinates $(-\frac{5}{2}, \frac{5\sqrt{3}}{2}, -1)$.

3. Determine the rectangular coordinates of the point with cylindrical coordinates $(3, \frac{\pi}{6}, 2)$.

4. Determine the rectangular coordinates of the point with cylindrical coordinates $(2, \frac{11\pi}{6}, -2)$.

XVIII-7 Given a rectangular equation, determine the equation in cylindrical coordinates, and vice versa.

1. Using cylindrical coordinates, determine the equation for $x^2 + y^2 = 64$.

2. Using cylindrical coordinates, determine the equation for $x^2 + y^2 - 8y = 0$.

3. The equation $r^2 \cos 2\theta = 6$ is in cylindrical coordinates. Determine the equation in rectangular coordinates.

4. The equation $r + z = 2$ is in cylindrical coordinates. Determine the equation in rectangular coordinates.

XVIII-8 Determine the spherical coordinates of a point with rectangular coordinates and vice versa.

1. Determine the spherical coordinates of the point whose rectangular coordinates are (1, 1, 1,).

2. Determine the spherical coordinates of the point whose rectangular coordinates are $(1, -\sqrt{3}, 2)$.

3. Determine the rectangular coordinates of the point whose spherical coordinates are $(2, \frac{\pi}{3}, \frac{\pi}{4})$.

4. Determine the rectangular coordinates of the point whose spherical coordinates are $(10, \frac{2\pi}{3}, \frac{\pi}{6})$.

**XVIII-9** Given a rectangular equation, determine the equation in spherical coordinates, and vice versa.

1. An equation in rectangular coordinates is $x^2 + y^2 + z^2 = 4$. What is the equation in spherical coordinates?

2. An equation in rectangular coordinates is $z = 9$. What is the equation in spherical coordinates?

3. An equation in spherical coordinates is $\rho = \frac{-1}{1 + \cos\phi}$. What is the equation in rectangular coordinates?

4. An equation in spherical coordinates is $\phi = \frac{\pi}{3}$. What is the equation in rectangular coordinates?

**Pre-Calculus**

**Unit XVIII**

## ANSWERS

### XVIII-1

1. $(x-3)^2 + (y-1)^2 + (z+2)^2 = 16$
2. $(x-4)^2 + (y+6)^2 + z^2 = 3$
3. $(x+2)^2 + (y-3)^2 + (z+5)^2 = 74$
4. $(x+1)^2 + (y+3)^2 + (z-5)^2 = 61$

### XVIII-2

See graphs.

### XVIII-3

See graphs.

### XVIII-4

1. a) elliptic paraboloid

   b) symmetric to yz - and xz - planes

   c) $\frac{x^2}{2} + \frac{y^2}{4} = 0$; $\frac{x^2}{2} = z$; $\frac{y^2}{4} = z$

   point parabola parabola

   d) All three intercepts are 0.

   e) See graphs.

2. a) elliptic hyperboloid of one sheet

   b) symmetric to xy, xz, and yz -planes

   c) $\frac{x^2}{8} - \frac{y^2}{9} = 1$ hyperbola

   $\frac{x^2}{8} + \frac{z^2}{18} = 1$ ellipse

   $\frac{z^2}{18} - \frac{y^2}{9} = 1$ hyperbola

   d) intercepts at $(\pm 2\sqrt{2}, 0, 0)$ $(0, 0, \pm 3\sqrt{2}))$. No y- intercepts

   e) See graphs.

3. a) ellipsoid

   b) symmetric to xy- xz- and yz planes

   c) $$\left.\begin{array}{l} \frac{x^2}{1} + \frac{y^2}{64} = 1 \\ \frac{x^2}{1} + \frac{z^2}{4} = 1 \\ \frac{y^2}{64} + \frac{z^2}{4} = 1 \end{array}\right\}\text{All are ellipses.}$$

   d) intercepts at $(\pm 1, 0, 0)$ $(0, \pm 8, 0)$ $(0, 0, \pm 4)$

   e) See graphs.

4. a) sphere, radius 1, center (1, 0, 0)

   b) symmetric to xz- and xy- planes

   c) $(x-1)^2 + y^2 = 1$ circle, center (1, 0, 0), radius 1

   $(x-1)^2 + z^2 = 1$ circle, center (1, 0, 0) radius 1

   $y^2 + z^2 = 1$ circle, center (0, 0, 0) radius 1

   d) intercepts at (0, 0, 0), (2, 0, 0); $(1, \pm 1, 0)$, and $(1, 0, \pm 1)$

   e) See graphs.

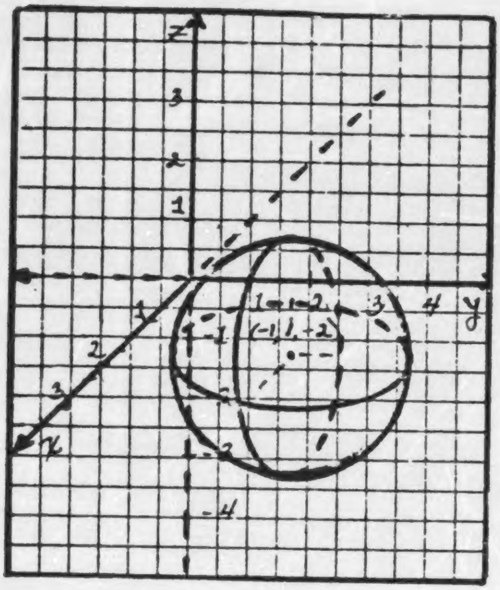
Pre-Calculus

Unit XVIII

ANSWERS

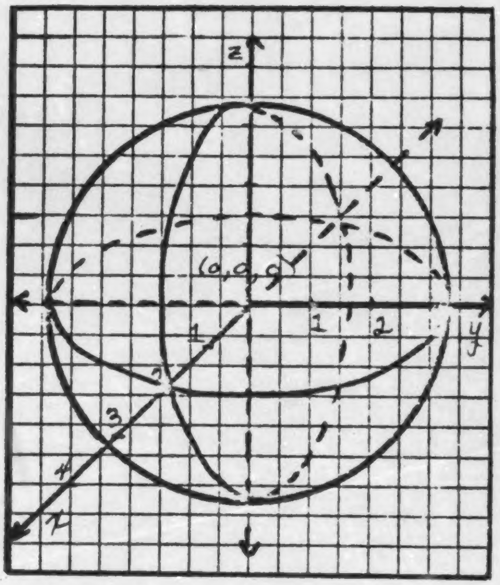
XVIII-2

1)



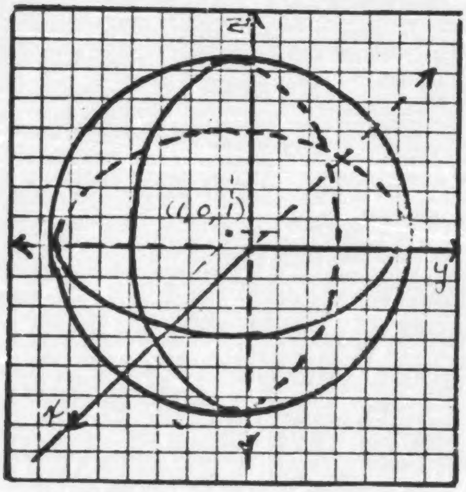
Center (-1, 1, -2)
Radius (2)

2)



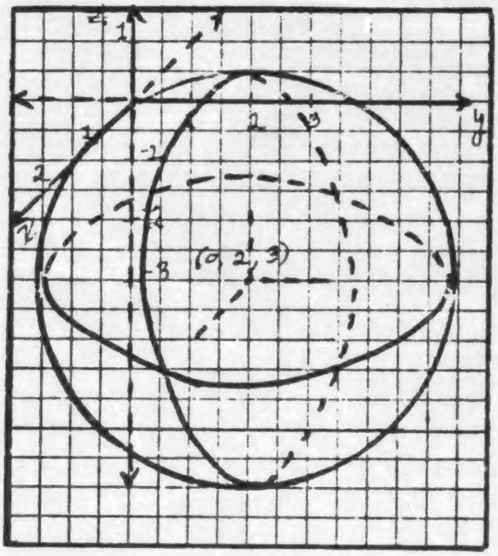
Center (0, 0, 0)
Radius ($2\sqrt{3}$)

3)



Center (1, 0, 1)
Radius (6)

4)



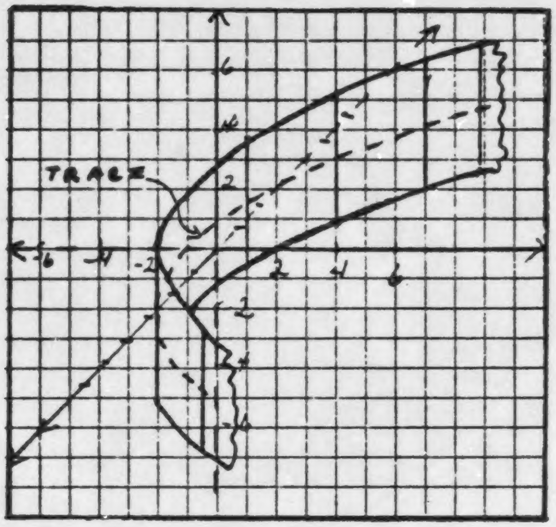
Center (0, 2, -3)
Radius ($\frac{7}{2}$)

Pre-Calculus

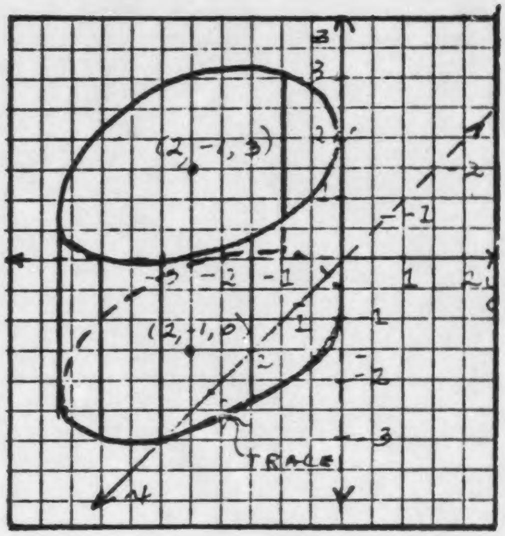
Unit XVIII
ANSWERS

XVIII-3

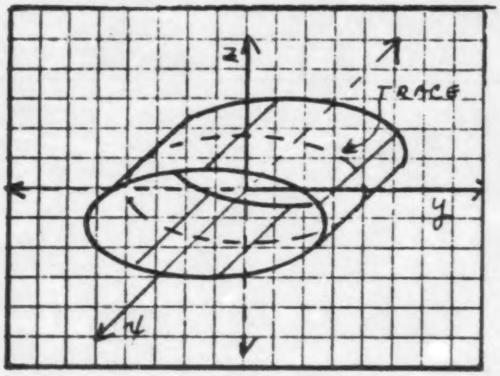
1)



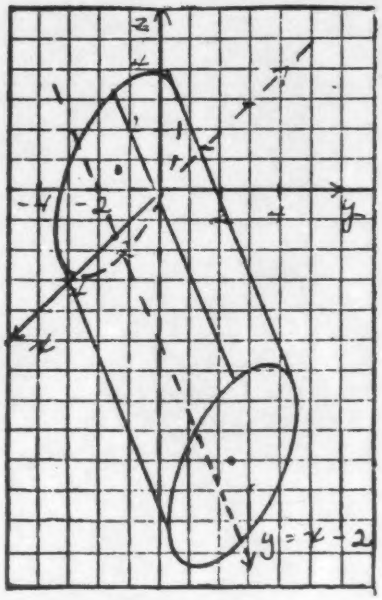
2)



3)



4)



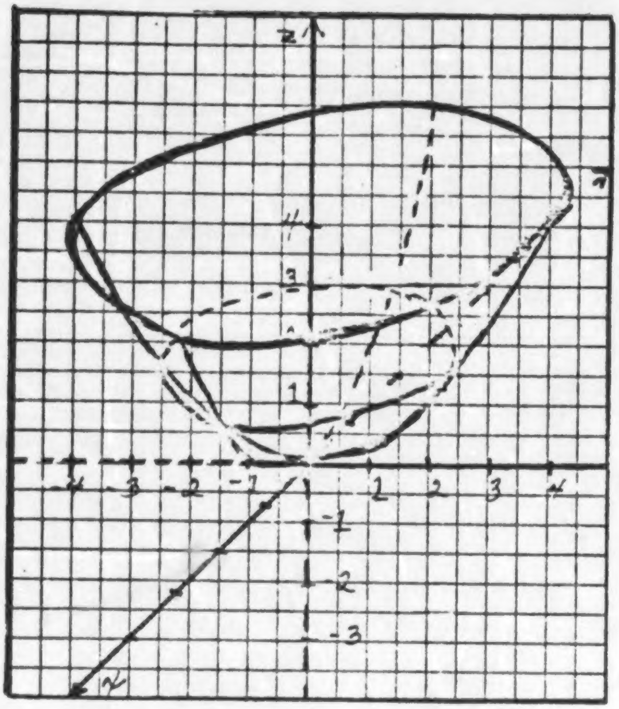
Pre-Calculus

Unit XVIII

ANSWERS

XVIII-4

1.



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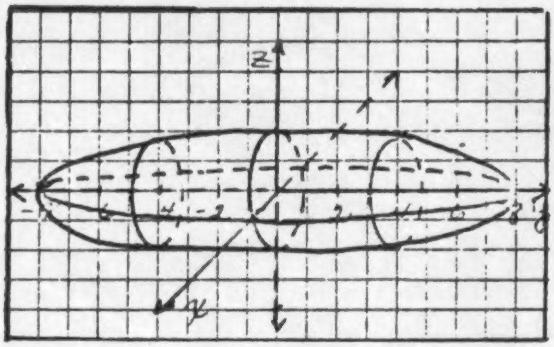
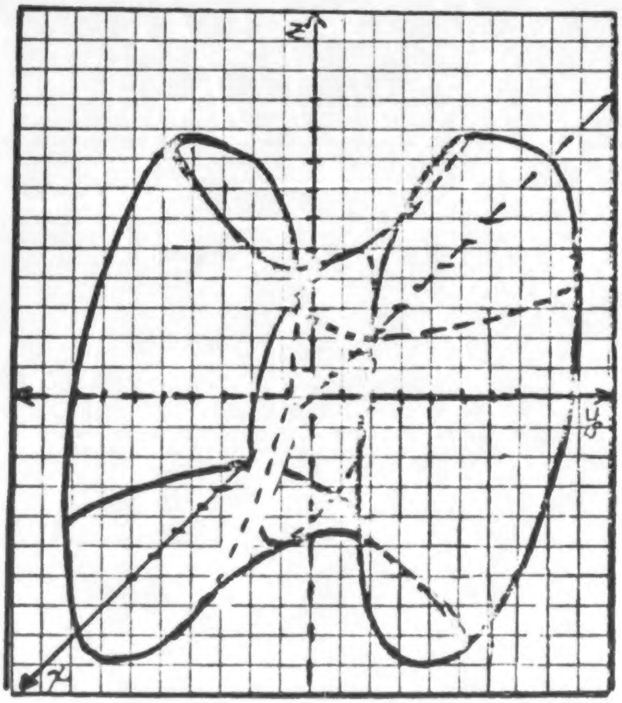


2.



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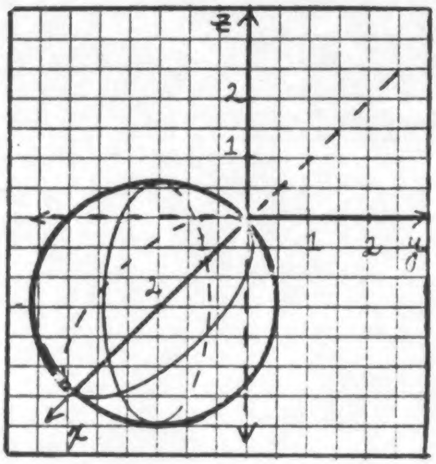


Pre-Calculus

Unit XVIII

## ANSWERS

### XVIII-5

1. a) $\frac{x^2}{4} + \frac{y^2}{4} = 1$ circle

$$\left.\begin{array}{l} 4y^2 - z^2 = 36 \\ 4x^2 - z^2 = 36 \end{array}\right\} \text{hyperbola}$$

b) z - axis

c) See graphs.

2. a) $2x^2 + y^2 - 4y = 0$ ellipse

$y^2 - 4y + 2z^2 = 0$ ellipse

$2x^2 + 2z^2 = 0$ point

b) y -axis

c) See graphs.

3. a) $4x^2 + 4y^2 = 16$ circle

$$\left.\begin{array}{l} 4x^2 + z^2 = 16 \\ 4y^2 + z^2 = 16 \end{array}\right\} \text{ellipse}$$

b) z -axis

c) See graphs.

4. a) $x^2 + 4y^2 = 0$ point

$x^2 - z^2 = 0$

$4y^2 - z^2 = 0$

b) z -axis

c) See graphs.

### XVIII-6

1. $(8, \pi, 6)$
2. $(5, \frac{2\pi}{3}, -1)$
3. $(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 2)$
4. $(\sqrt{3}, -1, -2)$

### XVIII-7

1. $r = 8$
2. $r = 8 \sin\theta$
3. $x^2 - y^2 = 6$
4. $x^2 + y^2 - z^2 + 4z = 4$

### XVIII-8

1. $(\sqrt{3}, \frac{\pi}{4}, \arccos \frac{1}{\sqrt{3}})$
2. $(2\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4})$
3. $(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2})$
4. $(-\frac{5}{2}, +\frac{5\sqrt{3}}{2}, 5\sqrt{3})$

### XVIII-9

1. $^2 = 4$
2. $\cos \phi = 9$
3. $x^2 + y^2 = 2z + 1$
4. $x^2 + y^2 - 3z^2 = 0$

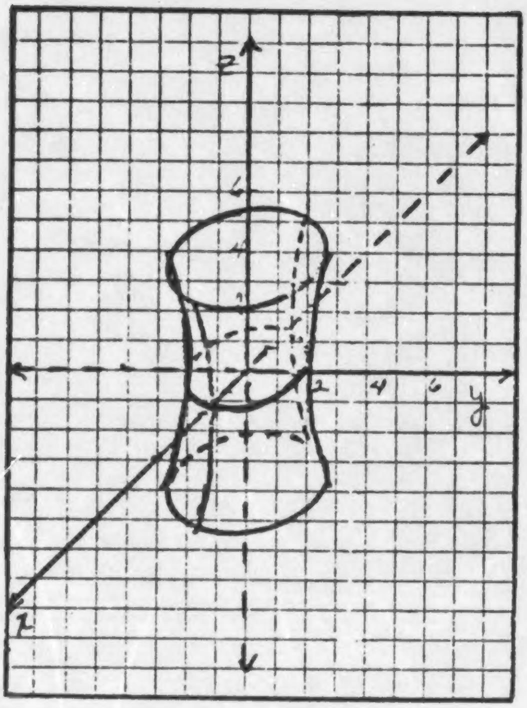
Pre-Calculus

Unit XVIII

**ANSWERS**

XVIII-5

1.



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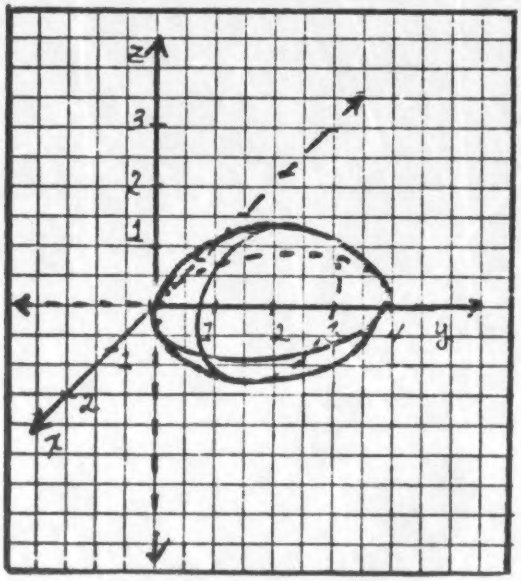

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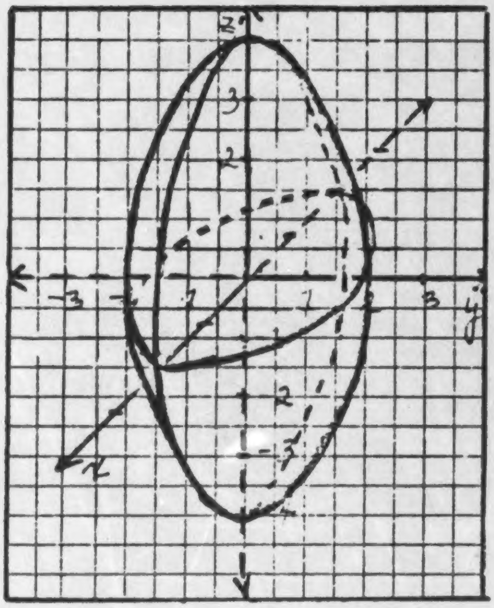

4.